ON SCALE FUNCTIONS OF SPECTRALLY NEGATIVE LÉVY PROCESSES WITH PHASE-TYPE JUMPS

MASAHIKO EGAMI AND KAZUTOSHI YAMAZAKI

ABSTRACT. We study the scale function of the spectrally negative phase-type Lévy process. Its scale function admits an analytical expression and so do a number of its fluctuation identities. Motivated by the fact that the class of phase-type distributions is dense in the class of all positive-valued distributions, we propose a new approach to approximating the scale function and the associated fluctuation identities for a general spectrally negative Lévy process. Numerical examples are provided to illustrate the effectiveness of the approximation method. The extension to the case with jumps of infinite activity is also discussed.

Key words: phase-type models; spectrally negative Lévy processes; scale functions; overshoot and undershoot
Mathematics Subject Classification (2000) : Primary: 60G51 Secondary: 60J75

1. INTRODUCTION

In the last decade, significant progress has been made regarding spectrally negative Lévy processes, and this is mainly due to the scale function. As can be seen in the work of, for example, [7, 25], a number of fluctuation identities concerning spectrally negative Lévy processes can be expressed in terms of scale functions. There are numerous applications in applied probability including optimal stopping, queuing, branching processes, insurance and credit risk. Despite these advances, a major obstacle still remains in putting these in practice because scale functions are in general known only up to their Laplace transforms, and only a few cases admit explicit expressions. Typically, one needs to rely on numerical Laplace inversion in order to approximate the scale function; see [24, 38].

In this paper, we propose a *phase-type (PH)-fitting* approach by using the scale function for the class of spectrally negative PH Lévy processes, or Lévy processes with negative PH-distributed jumps. Consider a continuous-time Markov chain with some initial distribution and state space consisting of a single absorbing state and a finite number of transient states. The PH distribution is the distribution of the time to absorption. The class of PH distributions includes, for example, the exponential, hyperexponential, Erlang, hyper-Erlang and Coxian distributions; see, e.g. Section 3 of [2]. It is known that the class of PH distributions is dense in the class of all positive-valued distributions, and consequently the scale function of any spectrally negative Lévy process can be approximated by those of PH Lévy processes.

First draft: May 1, 2010; this version: July 5, 2011.
M. Egami is in part supported by Grant-in-Aid for Scientific Research (B) No. 22330098 and (C) No. 20530340, Japan Society for the Promotion of Science. K. Yamazaki is in part supported by Grant-in-Aid for Young Scientists (B) No. 22710143, the Ministry of Education, Culture, Sports, Science and Technology, and by Grant-in-Aid for Scientific Research (B) No. 2271014, Japan Society for the Promotion of Science.

Our objective is to obtain the scale function of the spectrally negative PH Lévy process, and use it to approximate for a general case. We achieve its closed-form expression by taking advantage of the structure of its Wiener-Hopf factor as obtained in [3]. In order to verify that this is indeed a useful alternative to the Laplace inversion approach, we show convergence results via the continuity theorem and then conduct a series of numerical experiments.

One major advantage of this approach is that the approximated scale function is given as a function in a closed form, which enables one to analytically differentiate/integrate to obtain other fluctuation identities explicitly. The derivative of the scale function is used to identify, for example, the fluctuation of its reflected process and is commonly applied in insurance risk literature [5, 27, 30]. The integration of the scale function with respect to the Lévy measure is required when overshoots/undershoots are involved. On the other hand, these calculations must be conducted numerically when the scale function is approximated via numerical Laplace inversion; because the scale function increases exponentially and its derivative tends to explode near zero, minimizing numerical errors tends to be a difficult task.

Another advantage of the PH-fitting approach is that the Laplace transform of the PH distribution has an explicit expression. The Laplace inversion approach inverts the equality written in terms of the Laplace exponents which admit analytical expressions only in special cases (see (2.1)-(2.2) below). In other words, it contains two types of errors: 1) the approximation error caused while computing the Laplace exponent and 2) the error caused while inverting the Laplace transform. On the other hand, the PH-fitting approach only contains the PH-fitting error thanks to the closed-form Laplace transform of the PH distribution.

The PH-fitting approach is particularly efficient when the Lévy density is completely monotone because the fitting can be conducted by hyperexponential distributions (i.e. *hyperexponential-fitting*) and there exist algorithms that are guaranteed to converge (see, e.g., [15]). The class of Lévy processes with completely monotone Lévy density is rich. It includes compound-Poisson processes with long-tailed distributed jumps such as the Pareto, Weibull and Gamma distributions and other processes such as variance gamma [35, 36], CGMY [11], generalized hyperbolic [14] and normal inverse Gaussian [6] processes. The completely monotone assumption of the Lévy density is sometimes required, for example, in the optimal dividend problem [33, 34]. In order to give examples of how hyperexponential-fitting can be applied to compute fluctuation identities, we obtain the closed-form expressions of the overshoot/undershoot distributions at the first down-crossing time for the hyperexponential case, and use it to approximate those for the processes with Weibull/Pareto-type jumps.

The Lévy measure of the PH Lévy process is only limited to a finite measure. To complement this, we also consider the meromorphic Lévy process [22], whose Lévy measure may be infinite and is given as a countable sum of exponential functions. This generalizes the hyperexponential Lévy process as well as a number of Lévy processes such as Lamperti-stable processes [10, 13], hypergeometric processes [23, 26] and those in the β- and Θ-families [20, 21]. The corresponding scale function can be expressed explicitly as a *countable* sum of exponential functions, which can be effectively approximated by a *finite* sum with some analytical error bounds. We show numerically the effectiveness of the approximation procedure using examples with β-processes [20].

The rest of the paper is organized as follows. Section 2 studies the spectrally negative PH Lévy process and its scale function. We show via the continuity theorem that the PH-fitting approach can approximate the scale function for a general spectrally negative Lévy process. We further obtain, for the hyperexponential case, the joint distribution of the overshoot and undershoot at the first down-crossing time. Section 3 obtains the scale function

for the meromorphic Lévy process and its upper and lower bounds. Section 4 verifies the effectiveness of the PH-fitting approach through a series of numerical results. All proofs are given in the appendix.

2. Spectrally negative Phase-type Lévy processes

2.1. **Scale functions.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space hosting a *spectrally negative* Lévy process $X = \{X_t; t \geq 0\}$, $\mathbb{P}^x$ be the conditional probability under which $X_0 = x$, and $\mathbb{F} := \{\mathcal{F}_t : t \geq 0\}$ be the filtration generated by X. The process X is uniquely characterized by its *Laplace exponent*

$$\psi(s) := \log \mathbb{E}^0 \left[e^{sX_1} \right] = \hat{\mu} s + \frac{1}{2}\sigma^2 s^2 + \int_{-\infty}^0 (e^{sz} - 1 - sz 1_{\{z > -1\}}) \Pi(\mathrm{d}z), \quad s \in \mathbb{C} \tag{2.1}$$

where Π is a Lévy measure with the support $(-\infty, 0)$ and satisfies the integrability condition $\int_{(-\infty,0)} (1 \wedge z^2) \Pi(\mathrm{d}z) < \infty$. It has paths of bounded variation if and only if

$$\sigma = 0 \quad \text{and} \quad \int_{(-\infty,0)} (1 \wedge |z|) \Pi(\mathrm{d}z) < \infty;$$

see, for example, [25], Lemma 2.12. In this case, we can rewrite the Laplace exponent (2.1) by

$$\psi(s) = \mu s + \int_{-\infty}^0 (e^{sz} - 1) \Pi(\mathrm{d}z), \quad s \in \mathbb{C},$$

with

$$\mu := \hat{\mu} - \int_{-1}^0 z \Pi(\mathrm{d}z).$$

Fix $q \geq 0$ and any spectrally negative Lévy process with its Laplace exponent ψ. The scale function $W^{(q)} : \mathbb{R} \mapsto \mathbb{R}$ is a function whose Laplace transform is given by

$$\int_0^\infty e^{-sx} W^{(q)}(x) \mathrm{d}x = \frac{1}{\psi(s) - q}, \qquad s > \zeta_q \tag{2.2}$$

where

$$\zeta_q := \sup\{s \geq 0 : \psi(s) = q\}, \quad q \geq 0. \tag{2.3}$$

We assume $W^{(q)}(x) = 0$ on $(-\infty, 0)$.

Let us define the *first down-* and *up-crossing times*, respectively, by

$$\tau_a^- := \inf\{t \geq 0 : X_t < a\} \quad \text{and} \quad \tau_b^+ := \inf\{t \geq 0 : X_t > b\}, \quad a, b \in \mathbb{R}. \tag{2.4}$$

Then we have for any $x < b$

(2.5)

$$\mathbb{E}^x \left[e^{-q\tau_b^+} 1_{\{\tau_b^+ < \tau_0^-, \tau_b^+ < \infty\}} \right] = \frac{W^{(q)}(x)}{W^{(q)}(b)} \quad \text{and} \quad \mathbb{E}^x \left[e^{-q\tau_0^-} 1_{\{\tau_b^+ > \tau_0^-, \tau_0^- < \infty\}} \right] = Z^{(q)}(x) - Z^{(q)}(b) \frac{W^{(q)}(x)}{W^{(q)}(b)}$$

where

$$Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(y) \mathrm{d}y, \quad x \in \mathbb{R}. \tag{2.6}$$

Here, we disregard the case when X is a negative subordinator (or decreasing a.s.).

We also consider a version of the scale function $W_{\zeta_q} : \mathbb{R} \mapsto \mathbb{R}$ that satisfies

$$W^{(q)}(x) = e^{\zeta_q x} W_{\zeta_q}(x), \quad x \in \mathbb{R} \tag{2.7}$$

with its Laplace transform

$$\int_0^\infty e^{-sx} W_{\zeta_q}(x) \mathrm{d}x = \frac{1}{\psi(s+\zeta_q) - q}, \quad s > 0. \tag{2.8}$$

Suppose $\mathbb{P}_c$, for any given $c > 0$, is the probability measure defined by the Esscher transform

$$\left.\frac{\mathrm{d}\mathbb{P}_c}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = e^{cX_t - \psi(c)t}, \quad t \geq 0; \tag{2.9}$$

see page 78 of [25]. Then W_{ζ_q} under $\mathbb{P}_{\zeta_q}$ is equivalent to $W^{(0)}$ under $\mathbb{P}$. Furthermore, it is known that W_{ζ_q} is monotonically increasing and

$$W_{\zeta_q}(x) \nearrow (\psi'(\zeta_q))^{-1} \quad \text{as } x \to \infty, \tag{2.10}$$

which also implies that the scale function $W^{(q)}$ increases exponentially in x;

$$W^{(q)}(x) \sim \frac{e^{\zeta_q x}}{\psi'(\zeta_q)} \quad \text{as } x \to \infty, \tag{2.11}$$

except for the case $q = 0$ and $\psi'(\zeta_0) = 0$.

Regarding the smoothness of the scale function, if its jump distribution has no atoms, then $W^{(q)} \in C^1(0,\infty)$; if it has a Gaussian component ($\sigma > 0$), then $W^{(q)} \in C^2(0,\infty)$; see [12]. In particular, a stronger result holds for the completely monotone jump case. Recall that a density function f is called completely monotone if all the derivatives exist and, for every $n \geq 1$,

$$(-1)^n f^{(n)}(x) \geq 0, \quad x \geq 0,$$

where $f^{(n)}$ denotes the n^{th} derivative of f.

Lemma 2.1 (Loeffen [33])**.** *If the Lévy measure has a completely monotone density, W'_{ζ_q} is also completely monotone.*

Finally, the behavior in the neighborhood of zero is given as follows; see Lemmas 4.3-4.4 of [29].

Lemma 2.2. *For every $q \geq 0$, we have*

$$W^{(q)}(0) = \left\{ \begin{array}{ll} 0, & \textit{unbounded variation} \\ \frac{1}{\mu}, & \textit{bounded variation} \end{array} \right\} \quad \textit{and} \quad W^{(q)\prime}(0+) = \left\{ \begin{array}{ll} \frac{2}{\sigma^2}, & \sigma > 0 \\ \infty, & \sigma = 0 \textit{ and } \Pi(-\infty,0) = \infty \\ \frac{q+\Pi(-\infty,0)}{\mu^2}, & \textit{compound Poisson} \end{array} \right\}.$$

2.2. **Spectrally negative PH Lévy processes.** Consider a continuous-time Markov chain $Y = \{Y_t; t \geq 0\}$ with finite state space $\{1, \ldots, m\} \cup \{\Delta\}$ where $1, \ldots, m$ are transient and Δ is absorbing. Its initial distribution is given by a simplex $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_m]$ such that $\alpha_i = \mathbb{P}\{Y_0 = i\}$ for every $i = 1, \ldots, m$. The intensity matrix $\boldsymbol{Q}$ is partitioned into the m transient states and the absorbing state Δ, and is given by

$$\boldsymbol{Q} := \begin{bmatrix} \boldsymbol{T} & \boldsymbol{t} \\ \boldsymbol{0} & 0 \end{bmatrix}.$$

Here $\boldsymbol{T}$ is an $m \times m$-matrix called the PH-generator, and $\boldsymbol{t} = -\boldsymbol{T}\boldsymbol{1}$ where $\boldsymbol{1} = [1, \ldots, 1]'$. A distribution is called phase-type (PH) with representation $(m, \boldsymbol{\alpha}, \boldsymbol{T})$ if it is the distribution of the absorption time to Δ in the Markov chain described above. It is known that $\boldsymbol{T}$ is non-singular and thus invertible; see [1]. Its distribution and density functions are given, respectively, by

$$F(z) = 1 - \boldsymbol{\alpha} e^{\boldsymbol{T} z}\boldsymbol{1} \quad \text{and} \quad f(z) = \boldsymbol{\alpha} e^{\boldsymbol{T} z}\boldsymbol{t}, \quad z \geq 0.$$

Let $X = \{X_t; t \geq 0\}$ be a spectrally negative Lévy process of the form

$$X_t - X_0 = \mu t + \sigma B_t - \sum_{n=1}^{N_t} Z_n, \quad 0 \leq t < \infty, \tag{2.12}$$

for some $\mu \in \mathbb{R}$ and $\sigma \geq 0$. Here $B = \{B_t; t \geq 0\}$ is a standard Brownian motion, $N = \{N_t; t \geq 0\}$ is a Poisson process with arrival rate λ, and $Z = \{Z_n; n = 1, 2, \ldots\}$ is an i.i.d. sequence of PH-distributed random variables with representation $(m, \boldsymbol{\alpha}, \boldsymbol{T})$. These processes are assumed mutually independent. Its Laplace exponent is then

$$\psi(s) = \mu s + \frac{1}{2}\sigma^2 s^2 + \lambda \left(\boldsymbol{\alpha}(s\boldsymbol{I} - \boldsymbol{T})^{-1}\boldsymbol{t} - 1 \right), \tag{2.13}$$

which is analytic for every $s \in \mathbb{C}$ except for the eigenvalues of $\boldsymbol{T}$.

Disregarding the negative subordinator case, we consider the following two cases:

Case 1: when $\sigma > 0$ (i.e. X is of unbounded variation),

Case 2: when $\sigma = 0$ and $\mu > 0$ (i.e. X is a compound Poisson process).

Here, in Case 2, down-crossing of a threshold can occur only by jumps; see, for example, Chapter III of [7]. On the other hand, in Case 1, down-crossing can occur also by *creeping downward* (by the diffusion components). Due to this difference, the form of the scale function differs as we shall see.

Fix $q > 0$. Consider the *Cramér-Lundberg* equation

$$\psi(s) = q, \tag{2.14}$$

and define the set of (the absolute values of) *negative roots* and the set of *poles*:

$$\mathcal{I}_q := \{i : \psi(-\xi_{i,q}) = q \text{ and } \mathcal{R}(\xi_{i,q}) > 0\},$$
$$\mathcal{J}_q := \left\{ j : \frac{q}{q - \psi(-\eta_j)} = 0 \text{ and } \mathcal{R}(\eta_j) > 0 \right\}.$$

The elements in $\mathcal{I}_q$ and $\mathcal{J}_q$ may not be distinct, and, in this case, we take each as many times as its multiplicity. By Lemma 1 of [3], we have

$$|\mathcal{I}_q| = \begin{cases} |\mathcal{J}_q| + 1, & \text{for Case } 1, \\ |\mathcal{J}_q|, & \text{for Case } 2. \end{cases}$$

In particular, if the representation is minimal (see [3]), we have $|\mathcal{J}_q| = m$.

Let $\mathbf{e}_q$ be an independent exponential random variable with parameter q and denote the *running maximum* and *minimum*, respectively, by

$$\overline{X}_t = \sup_{0 \leq s \leq t} X_s \quad \text{and} \quad \underline{X}_t = \inf_{0 \leq s \leq t} X_s, \quad t \geq 0.$$

The *Wiener-Hopf factorization* states that $q/(q-\psi(s)) = \varphi_q^+(s)\varphi_q^-(s)$ for every $s \in \mathbb{C}$ such that $\mathcal{R}(s) = 0$, with the *Wiener-Hopf factors*

$$\varphi_q^-(s) := \mathbb{E}\left[\exp(s\underline{X}_{\mathbf{e}_q})\right] \quad \text{and} \quad \varphi_q^+(s) := \mathbb{E}\left[\exp(s\overline{X}_{\mathbf{e}_q})\right] \tag{2.15}$$

that are analytic for s with $\mathcal{R}(s) > 0$ and $\mathcal{R}(s) < 0$, respectively. By Lemma 1 of [3], we have, for every s such that $\mathcal{R}(s) > 0$,

$$\varphi_q^-(s) = \frac{\prod_{j \in \mathcal{J}_q}(s+\eta_j)}{\prod_{j \in \mathcal{J}_q}\eta_j} \frac{\prod_{i \in \mathcal{I}_q}\xi_{i,q}}{\prod_{i \in \mathcal{I}_q}(s+\xi_{i,q})}, \tag{2.16}$$

from which we can obtain the distribution of $\underline{X}_{\mathbf{e}_q}$ by the Laplace inverse via partial fraction expansion.

As in Remark 4 of [3], let n denote the number of different roots in $\mathcal{I}_q$ and m_i denote the multiplicity of a root $\xi_{i,q}$ for $i = 1, \ldots, n$. Then we have

$$\mathbb{P}\left\{-\underline{X}_{\mathbf{e}_q} \in \mathrm{d}x\right\} = \sum_{i=1}^{n}\sum_{k=1}^{m_i} A_{i,q}^{(k)}\xi_{i,q}\frac{(\xi_{i,q}x)^{k-1}}{(k-1)!}e^{-\xi_{i,q}x}\mathrm{d}x, \quad x > 0 \tag{2.17}$$

where

$$A_{i,q}^{(k)} := \frac{1}{(m_i-k)!}\frac{\partial^{m_i-k}}{\partial s^{m_i-k}}\frac{\varphi_q^-(s)(s+\xi_{i,q})^{m_i}}{\xi_{i,q}^k}\bigg|_{s=-\xi_{i,q}}.$$

Notice that this can be simplified significantly when all the roots in $\mathcal{I}_q$ are distinct.

2.3. **Scale functions for spectrally negative PH Lévy processes.** We obtain the scale function focusing on the case $q > 0$. The scale function when $q = 0$ and X drifts to $+\infty$ can be derived by using $W^{(0)}(x) = \mathbb{P}^x\left\{\underline{X}_\infty \geq 0\right\}/\psi'(0)$ and the ruin probability (19) of [3] by taking $q \to 0$; Kyprianou and Palmowski [27] briefly stated the scale function when $q = 0$, X drifts to $+\infty$ and all the roots in $\mathcal{I}_q$ are distinct. The case $q = 0$ and X drifts to $-\infty$ can be obtained indirectly by change of measure (2.9); in this case, $W^{(0)}(x) = e^{\zeta_0 x}W_{\zeta_0}(x)$ with $\zeta_0 > 0$ and X drifts to $+\infty$ under $\mathbb{P}_{\zeta_q}$.

Before obtaining the scale function, we shall first represent the positive root ζ_q (2.3) in terms of the negative roots $\{\xi_{i,q};\ i \in \mathcal{I}_q\}$. Let us define

$$\varrho_q := \sum_{i=1}^{n} A_{i,q}^{(1)}\xi_{i,q}, \quad q > 0, \tag{2.18}$$

and by Lemma 2.2

$$\theta := -\zeta_q W^{(q)}(0) + W^{(q)\prime}(0+) = \left\{\begin{array}{ll} \frac{2}{\sigma^2}, & \text{for Case 1} \\ -\frac{\zeta_q}{\mu} + \frac{q+\lambda}{\mu^2}, & \text{for Case 2} \end{array}\right\}. \tag{2.19}$$

Lemma 2.3. *For every $q > 0$, we have*

$$\frac{\zeta_q}{q} = \frac{\theta}{\varrho_q}.$$

We now obtain the version of the scale function $W_{\zeta_q}(\cdot)$. In the lemma below, $W_{\zeta_q}(0) = W^{(q)}(0)$ is either 0 or $\frac{1}{\mu}$ depending on if it is Case 1 or 2; see Lemma 2.2.

Lemma 2.4. *For every $q > 0$, we have*

$$W_{\zeta_q}(x) - W_{\zeta_q}(0) = \frac{\zeta_q}{q} \sum_{i=1}^{n} \sum_{k=1}^{m_i} A_{i,q}^{(k)} \left(\frac{\xi_{i,q}}{\zeta_q + \xi_{i,q}} \right)^k \left[1 - e^{-(\zeta_q + \xi_{i,q})x} \sum_{j=0}^{k-1} \frac{((\zeta_q + \xi_{i,q})x)^j}{j!} \right], \quad x \geq 0.$$

Lemmas 2.2 and 2.4 and (2.7) show the following. By Lemma 2.3, these expressions can also be rewritten in a different way.

Proposition 2.1. *Fix $q > 0$ and $x \geq 0$.*

(1) *For Case 1,*

$$W^{(q)}(x) = \frac{\zeta_q}{q} \sum_{i=1}^{n} \sum_{k=1}^{m_i} A_{i,q}^{(k)} \left(\frac{\xi_{i,q}}{\zeta_q + \xi_{i,q}} \right)^k \left[e^{\zeta_q x} - e^{-\xi_{i,q} x} \sum_{j=0}^{k-1} \frac{((\zeta_q + \xi_{i,q})x)^j}{j!} \right].$$

(2) *For Case 2,*

$$W^{(q)}(x) = \frac{\zeta_q}{q} \sum_{i=1}^{n} \sum_{k=1}^{m_i} A_{i,q}^{(k)} \left(\frac{\xi_{i,q}}{\zeta_q + \xi_{i,q}} \right)^k \left[e^{\zeta_q x} - e^{-\xi_{i,q} x} \sum_{j=0}^{k-1} \frac{((\zeta_q + \xi_{i,q})x)^j}{j!} \right] + \frac{1}{\mu} e^{\zeta_q x}.$$

The scale functions obtained above are infinitely differentiable. In particular, the first derivative becomes

$$W^{(q)\prime}(x) = \frac{\zeta_q}{q} \sum_{i=1}^{n} \sum_{k=1}^{m_i} A_{i,q}^{(k)} \left(\frac{\xi_{i,q}}{\zeta_q + \xi_{i,q}} \right)^k \left[\zeta_q e^{\zeta_q x} + \xi_{i,q} e^{-\xi_{i,q} x} \frac{((\zeta_q + \xi_{i,q})x)^{k-1}}{(k-1)!} - \zeta_q e^{-\xi_{i,q} x} \sum_{j=0}^{k-2} \frac{((\zeta_q + \xi_{i,q})x)^j}{j!} \right]$$

for Case 1 and

$$W^{(q)\prime}(x) = \frac{\zeta_q}{q} \sum_{i=1}^{n} \sum_{k=1}^{m_i} A_{i,q}^{(k)} \left(\frac{\xi_{i,q}}{\zeta_q + \xi_{i,q}} \right)^k \times \left[\zeta_q e^{\zeta_q x} + \xi_{i,q} e^{-\xi_{i,q} x} \frac{((\zeta_q + \xi_{i,q})x)^{k-1}}{(k-1)!} - \zeta_q e^{-\xi_{i,q} x} \sum_{j=0}^{k-2} \frac{((\zeta_q + \xi_{i,q})x)^j}{j!} \right] + \frac{1}{\mu} \zeta_q e^{\zeta_q x}$$

for Case 2.

When all the roots in $\mathcal{I}_q$ are distinct, the scale functions above can be simplified and have nice properties as discussed in the following corollary.

Corollary 2.1. *Suppose all the roots in $\mathcal{I}_q$ are distinct.*

(1) *The scale function can be simplified to*

$$W^{(q)}(x) = \sum_{i \in \mathcal{I}_q} C_{i,q} \left[e^{\zeta_q x} - e^{-\xi_{i,q} x} \right],$$
$$W^{(q)}(x) = \sum_{i \in \mathcal{I}_q} C_{i,q} \left[e^{\zeta_q x} - e^{-\xi_{i,q} x} \right] + \frac{1}{\mu} e^{\zeta_q x}, \tag{2.20}$$

for Case 1 and 2, respectively where

$$C_{i,q} := \frac{\zeta_q}{q} \frac{\xi_{i,q} A^{(1)}_{i,q}}{\zeta_q + \xi_{i,q}}, \quad i \in \mathcal{I}_q.$$

(2) $W^{(q)\prime}$ *is convex.*

(3) W'_{ζ_q} *is completely monotone.*

In the lemma above, by (2.10)-(2.11), we must have

$$\frac{1}{\psi'(\zeta_q)} = \sum_{i \in \mathcal{I}_q} C_{i,q} \quad \text{and} \quad \frac{1}{\psi'(\zeta_q)} = \sum_{i \in \mathcal{I}_q} C_{i,q} + \frac{1}{\mu}, \tag{2.21}$$

respectively for Case 1 and 2.

Example 2.1 (Hyperexponential Case)**.** *As an important example where all the roots in $\mathcal{I}_q$ are distinct, we consider the case where Z has a hyperexponential distribution with density function*

$$f(z) = \sum_{j=1}^{m} p_j \eta_j e^{-\eta_j z}, \quad z \geq 0,$$

for some $0 < \eta_1 < \cdots < \eta_m < \infty$. Its Laplace exponent (2.1) is then

$$\psi(s) = \mu s + \frac{1}{2} \sigma^2 s^2 - \lambda \sum_{j=1}^{m} p_j \frac{s}{\eta_j + s}. \tag{2.22}$$

Notice in this case that $-\eta_1, \ldots, -\eta_m$ are the poles of the Laplace exponent. Furthermore, all the roots in $\mathcal{I}_q$ are distinct and satisfy the following interlacing condition for every $q > 0$:

(1) *when $\sigma > 0$, there are $m+1$ roots $-\xi_{1,q}, \ldots, -\xi_{m+1,q}$ such that*

$$0 < \xi_{1,q} < \eta_1 < \xi_{2,q} < \cdots < \eta_m < \xi_{m+1,q} < \infty;$$

(2) *when $\sigma = 0$ and $\mu > 0$, there are m roots $-\xi_{1,q}, \ldots, -\xi_{m,q}$ such that*

$$0 < \xi_{1,q} < \eta_1 < \xi_{2,q} < \cdots < \xi_{m,q} < \eta_m < \infty.$$

The class of hyperexponential distributions is important as it is dense in the class of all positive-valued distributions with completely monotone densities.

2.4. **Approximation of the scale function of a general spectrally negative Lévy process.** The scale function obtained in Proposition 2.1 can be used to approximate the scale function of a general spectrally negative Lévy process.

By Proposition 1 of [3], there exists, for any spectrally negative Lévy process X, a sequence of spectrally negative PH Lévy processes $X^{(n)}$ converging to X in $D[0,\infty)$. This is equivalent to saying that $X_1^{(n)} \to X_1$ in distribution by [18], Corollary VII 3.6; see also [37]. Suppose ψ_n (resp. ψ), $\zeta_{q,n}$ (ζ_q) and $W_n^{(q)}/W_{\zeta_q,n}$ ($W^{(q)}/W_{\zeta_q}$) are the Laplace exponent, the positive root (2.3) and the scale function of $X^{(n)}$ (X), respectively. Because $W^{(q)}$ is an increasing function, the measure $W^{(q)}(\mathrm{d}x)$ associated with the distribution of $W^{(q)}(0,x]$ is well-defined and we obtain as in page 218 of [25],

$$\int_{[0,\infty)} e^{-\beta x} W^{(q)}(\mathrm{d}x) = \frac{\beta}{\psi(\beta)-q}. \tag{2.23}$$

Because these processes are spectrally negative and ψ is continuous on $[0,\infty)$, we have, by the continuity theorem, $\psi_n(\beta) \to \psi(\beta)$ for every $\beta > 0$. Now in view of (2.23), the convergence of the scale function holds by the continuity of the scale function and the continuity theorem; see [16], Theorem 2a, XIII.1. In other words, $W_n^{(q)}(x) \to W^{(q)}(x)$ and $W_{\zeta_q,n}(x) \to W_{\zeta_q}(x)$ as $n \uparrow \infty$ for every $x \geq 0$.

The smoothness and monotonicity properties of the scale function can be additionally used to obtain stronger results. The scale functions in Proposition 2.1 are in $C^\infty(0,\infty)$. In addition, when all the roots of $\mathcal{I}_q$ are different, its first derivative W'_{ζ_q} is completely monotone as discussed in Corollary 2.1.

Suppose that it is in $C^2(0,\infty)$ (which holds, for example, when $\sigma > 0$ by [12]), $W'_{\zeta_q}(0+) < \infty$ (i.e., $\sigma > 0$ or $\Pi(-\infty,0) < \infty$) and $W''_{\zeta_q}(x) \leq 0$ for every $x \geq 0$. Because $W'_{\zeta_q}(x) \xrightarrow{x\uparrow\infty} 0$, we see that

$$F(x) := (W'_{\zeta_q}(0+))^{-1} \int_0^x |W''_{\zeta_q}(y)| \mathrm{d}y$$

is a probability distribution and

$$\int_0^\infty e^{-\beta x} F(\mathrm{d}x) = (W'_{\zeta_q}(0+))^{-1} \left[-\frac{\beta^2}{\psi(\beta+\zeta_q)-q} + \beta W_{\zeta_q}(0) + W'_{\zeta_q}(0+) \right], \quad \beta > 0.$$

Therefore, noting that $W^{(q)\prime}(x) = \zeta_q W^{(q)}(x) + e^{\zeta_q x} W'_{\zeta_q}(x)$ and assuming that the convergent sequence $W_{\zeta_q,n}(x)$ has the same property, we can obtain by the continuity theorem

$$W'_{\zeta_q,n}(x) \xrightarrow{n\uparrow\infty} W'_{\zeta_q}(x) \quad \text{and} \quad W_n^{(q)\prime}(x) \xrightarrow{n\uparrow\infty} W^{(q)\prime}(x), \quad x \geq 0.$$

The key assumption of the negativity of W''_{ζ_q} holds, for example, for the completely monotone jump case because W'_{ζ_q} is completely monotone by [33]. We can also choose the sequence $W'_{\zeta_q,n}$ completely monotone in view of Corollary 2.1 because approximation can be done via hyperexponential distributions. In fact, it also means that W_{ζ_q} is $C^\infty(0,\infty)$ and the convergence of higher derivatives can be pursued. Even for a general jump distribution, the negativity of W''_{ζ_q} is a reasonable assumption in view of the numerical plots given by [38].

2.5. **Obtaining overshoot and undershoot distributions.** We conclude this section by giving an example how we can apply the explicit expression of the scale function obtained above. We consider the joint distribution of

overshoot and undershoot (with discounting):

$$h_q(x; A, B) := \mathbb{E}^x \left[e^{-q\tau_0^-} 1_{\{X_{\tau_0^- -} \in B, X_{\tau_0^-} \in A, \tau_0^- < \infty\}} \right], \quad B \in \mathcal{B}(0, \infty), \ A \in \mathcal{B}(-\infty, 0), \tag{2.24}$$

focusing on the hyperexponential case addressed in Example 2.1. Our objective is to show an example where the PH (hyperexponential)-fitting can be applied; we focus on this rather simple example because we later evaluate it as an approximation tool by comparing the result with the simulated expectation. The results obtained here can be easily extended to more general fluctuation identities; see e.g. [24].

We assume that X is a hyperexponential Lévy process and $q > 0$. The joint distribution (2.24) can be easily obtained via the resolvent measure and the compensation formula.

Lemma 2.5. *For all $B \in \mathcal{B}(0, \infty)$ and $A \in \mathcal{B}(-\infty, 0)$,*

$$h_q(x; A, B) = \int_0^\infty \overline{\Pi}(\mathrm{d}u) \left\{ W^{(q)}(x) \int_{B \cap (A+u)} e^{-\zeta_q y} \mathrm{d}y - \int_{B \cap (A+u)} \mathrm{d}y W^{(q)}(x - y) \right\},$$

where $\overline{\Pi}(\mathrm{d}u) = \lambda \sum_{j=1}^m p_j \eta_j e^{-\eta_j u} \mathrm{d}u$ for all $u \in (0, \infty)$.

After integrating with respect to $\overline{\Pi}$ and assume A and B are open intervals, we obtain the following.

Proposition 2.2. *Suppose $B = (\underline{b}, \overline{b})$ and $A = (-\overline{a}, -\underline{a})$ for some $0 \leq \underline{a} \leq \overline{a}$ and $0 \leq \underline{b} \leq \overline{b}$. Then*

$$h_q(x; A, B) = \lambda \sum_{j=1}^m p_j (e^{-\eta_j \underline{a}} - e^{-\eta_j \overline{a}}) \kappa_{j,q}(x; B)$$

where, for each $1 \leq j \leq m$,

$$\begin{aligned} \kappa_{j,q}(x; B) &:= \frac{e^{\zeta_q x}}{\psi'(\zeta_q)(\eta_j + \zeta_q)} \left(e^{-(\eta_j + \zeta_q)(\underline{b} \vee x)} - e^{-(\eta_j + \zeta_q)(\overline{b} \vee x)} \right) \\ &+ \sum_{i \in \mathcal{I}_q} C_{i,q} e^{-\xi_{i,q} x} \left[\frac{1}{(\eta_j - \xi_{i,q})} \left(e^{-(\eta_j - \xi_{i,q})(\underline{b} \wedge x)} - e^{-(\eta_j - \xi_{i,q})(\overline{b} \wedge x)} \right) - \frac{1}{(\eta_j + \zeta_q)} \left(e^{-(\eta_j + \zeta_q)\underline{b}} - e^{-(\eta_j + \zeta_q)\overline{b}} \right) \right]. \end{aligned}$$

By differentiating the above, we obtain the density:

$$\begin{aligned} \mathbb{E}^x \left[e^{-q\tau_0^-} 1_{\{-X_{\tau_0^-} \in \mathrm{d}a, X_{\tau_0^- -} \in B, \tau_0^- < \infty\}} \right] &= \lambda \sum_{j=1}^m p_j \eta_j e^{-\eta_j a} \kappa_{j,q}(x; B) \\ \mathbb{E}^x \left[e^{-q\tau_0^-} 1_{\{X_{\tau_0^- -} \in \mathrm{d}b, X_{\tau_0^-} \in A, \tau_0^- < \infty\}} \right] &= \lambda \sum_{j=1}^m p_j (e^{-\eta_j \underline{a}} - e^{-\eta_j \overline{a}}) \\ &\times \left\{ \begin{array}{ll} \sum_{i \in \mathcal{I}_q} C_{i,q} e^{-\xi_{i,q} x} \left(e^{-(\eta_j - \xi_{i,q}) b} - e^{-(\eta_j + \zeta_q) b} \right), & b < x \\ \frac{1}{\psi'(\zeta_q)} e^{\zeta_q x - (\eta_j + \zeta_q) b} - \sum_{i \in \mathcal{I}_q} C_{i,q} e^{-(\xi_{i,q} x + (\eta_j + \zeta_q) b)}, & b \geq x \end{array} \right\}. \end{aligned}$$

As a special case, the overshoot (resp. undershoot) density becomes, by setting $B = (0, \infty)$ $(A = (-\infty, 0))$,

(2.25)

$$\mathbb{E}^x \left[e^{-q\tau_0^-} 1_{\{-X_{\tau_0^-} \in \mathrm{d}a,\, \tau_0^- < \infty\}} \right] = \lambda \sum_{j=1}^m p_j \eta_j e^{-\eta_j a} \kappa_{j,q}(x; (0,\infty)),$$

$$\mathbb{E}^x \left[e^{-q\tau_0^-} 1_{\{X_{\tau_0^-_-} \in \mathrm{d}b,\, \tau_0^- < \infty\}} \right] = \begin{cases} \lambda \sum_{j=1}^m p_j \sum_{i \in \mathcal{I}_q} C_{i,q} e^{-\xi_{i,q} x} \left(e^{-(\eta_j - \xi_{i,q})b} - e^{-(\eta_j + \zeta_q)b} \right), & b < x, \\ \lambda \sum_{j=1}^m p_j \left[\frac{1}{\psi'(\zeta_q)} e^{\zeta_q x - (\eta_j + \zeta_q)b} - \sum_{i \in \mathcal{I}_q} C_{i,q} e^{-(\xi_{i,q} x + (\eta_j + \zeta_q)b)} \right], & b \geq x, \end{cases}$$

where

$$\kappa_{j,q}(x; (0,\infty)) = \frac{1}{\psi'(\zeta_q)} \frac{1}{(\eta_j + \zeta_q)} e^{-\eta_j x} + \sum_{i \in \mathcal{I}_q} C_{i,q} \left[\frac{1}{(\eta_j - \xi_{i,q})} \left(e^{-\xi_{i,q} x} - e^{-\eta_j x} \right) - e^{-\xi_{i,q} x} \frac{1}{(\eta_j + \zeta_q)} \right].$$

Remark 2.1. *Based on our numerical experiments, it can be conjectured that*

$$\frac{1}{\psi'(\zeta_q)} \frac{1}{(\eta_j + \zeta_q)} = \sum_{i \in \mathcal{I}_q} C_{i,q} \frac{1}{(\eta_j - \xi_{i,q})}, \quad j = 1, \ldots, m,$$

and consequently

$$\kappa_{j,q}(x; (0,\infty)) = \sum_{i \in \mathcal{I}_q} C_{i,q} \left(\frac{1}{(\eta_j - \xi_{i,q})} - \frac{1}{(\eta_j + \zeta_q)} \right) e^{-\xi_{i,q} x}.$$

3. Scale functions for Meromorphic Lévy processes

In this section, we consider another class of spectrally negative Lévy processes called meromorphic Lévy processes. We obtain their scale functions and use these as approximation tools for a general spectrally negative Lévy process with a completely monotone Lévy measure. Similarly to the approach applied in the last section, we obtain the scale function using its Wiener-Hopf factorization. It has a form expressed as an countable sum of exponential functions which can be bounded efficiently by finite sums.

3.1. **Meromorphic Lévy processes.** The following is due to [22], Definition 1.

Definition 3.1 (spectrally negative meromorphic Lévy process)**.** *A spectrally negative Lévy process X is said to be meromorphic if the following conditions hold.*

(1) *The Laplace exponent $\psi(s)$ (2.1) has a countable set of real negative poles.*
(2) *For every $q \geq 0$, the Cramér-Lundberg equation (2.14) has a countable set of real negative roots.*
(3) *Let $\{\eta_k; k \geq 1\}$ and $\{\xi_{k,q}; k \geq 1\}$, respectively, be the sets of the absolute values of the poles and the negative roots of (2.14) for fixed $q \geq 0$. Then it satisfies the following interlacing conditions:*

$$\cdots < -\eta_k < -\xi_{k,q} < \cdots < -\eta_2 < -\xi_{2,q} < -\eta_1 < -\xi_{1,q} < 0.$$

(4) *There exists $\alpha > \frac{1}{2}$ such that $\eta_k \sim ck^\alpha$ as $k \to \infty$.*
(5) *The Wiener-Hopf factor (2.15) is expressed as convergent infinite products*

$$\varphi_q^-(s) = \prod_{k=1}^\infty \frac{(s + \eta_k)}{\eta_k} \frac{\xi_{k,q}}{(s + \xi_{k,q})}. \tag{3.1}$$

The class of Meromorphic Lévy processes complements the class of PH Lévy processes described in the previous section because it also contains those with jumps of infinite activity. As noted by Corollary 3 of [22], the property (3) in Definition 4.1 is equivalent to the condition that the Lévy density has the form

$$\pi(z) = \sum_{j=1}^{\infty} \alpha_j \eta_j e^{-\eta_j |z|} 1_{\{z<0\}}, \quad z \in \mathbb{R}.$$

This can be seen as an extension to the hyperexponential case as described in Example 2.1. For more details, see [22].

The Wiener-Hopf factor (3.1) for this process is again a rational function as in (2.16) for the PH case. Therefore, this can be inverted again by partial fraction decomposition, and we have as in Corollary 1 of [22]

$$\mathbb{P}\left\{-\underline{X}_{\mathbf{e}_q} \in \mathrm{d}x\right\} = \sum_{i=1}^{\infty} A_{i,q} \xi_{i,q} e^{-\xi_{i,q} x} \mathrm{d}x, \quad x > 0 \tag{3.2}$$

where

$$A_{i,q} := \frac{s + \xi_{i,q}}{\xi_{i,q}} \varphi_q^-(s) \Big|_{s=-\xi_{i,q}} = \left(1 - \frac{\xi_{i,q}}{\eta_i}\right) \prod_{j \neq i} \frac{1 - \frac{\xi_{i,q}}{\eta_j}}{1 - \frac{\xi_{i,q}}{\xi_{j,q}}}, \quad i \geq 1.$$

Notice by the interlacing condition that $A_{i,q} > 0$ for every $i \geq 1$.

3.2. **Scale functions for meromorphic Lévy processes.** We now obtain the scale function for the meromorphic Lévy process. We omit the proof because it is similar to the PH case; see Appendix A.1.

Lemma 3.1. *For every $q > 0$, we have*

$$W_{\zeta_q}(x) - W_{\zeta_q}(0) = \sum_{i=1}^{\infty} C_{i,q} \left[1 - e^{-(\zeta_q + \xi_{i,q})x}\right], \quad x \geq 0$$

where

$$C_{i,q} := \frac{\zeta_q}{q} \frac{\xi_{i,q} A_{i,q}}{\zeta_q + \xi_{i,q}}, \quad i \geq 1. \tag{3.3}$$

By (2.10) and Lemma 3.1, we have, by taking the limit,

$$\gamma_q := \sum_{i=1}^{\infty} C_{i,q} = (\psi'(\zeta_q))^{-1} - W_{\zeta_q}(0) < \infty. \tag{3.4}$$

The scale function can be therefore obtained by Lemma 3.1 and (3.4).

Proposition 3.1. *For every $q > 0$, we have*

$$W^{(q)}(x) = \sum_{i=1}^{\infty} C_{i,q} \left[e^{\zeta_q x} - e^{-\xi_{i,q} x}\right] + W_{\zeta_q}(0) e^{\zeta_q x} = (\psi'(\zeta_q))^{-1} e^{\zeta_q x} - \sum_{i=1}^{\infty} C_{i,q} e^{-\xi_{i,q} x}, \quad x \geq 0. \tag{3.5}$$

By the dominated convergence theorem thanks to the non-negativity of $C_{i,q}$ and convexity of the exponential function, we have, for every $q > 0$ and $x > 0$,

$$W^{(q)\prime}(x) = (\psi'(\zeta_q))^{-1}\zeta_q e^{\zeta_q x} + \sum_{i=1}^{\infty} C_{i,q}\xi_{i,q}e^{-\xi_{i,q}x}.$$

By adjusting slightly the proof of Lemma 2.3 (given in Appendix A.1), we have the following.

Lemma 3.2. (1) *The following two statements are equivalent:*

(a) $\sigma = 0$ *and* $\Pi(-\infty, 0) = \infty$,

(b) $\sum_{i=1}^{\infty} A_{i,q}\xi_{i,q} = \infty$.

(2) *Suppose* $\sigma > 0$ *or* $\Pi(-\infty, 0) < \infty$. *Then, for every* $q > 0$*, we have*

$$\frac{\zeta_q}{q} = \theta \left(\sum_{i=1}^{\infty} A_{i,q}\xi_{i,q} \right)^{-1} \tag{3.6}$$

where

$$\theta := -\zeta_q W^{(q)}(0) + W^{(q)\prime}(0+) = \left\{ \begin{array}{ll} \frac{2}{\sigma^2}, & \text{when } \sigma > 0 \\ -\frac{\zeta_q}{\mu} + \frac{q+\Pi(-\infty,0)}{\mu^2}, & \text{when } \sigma = 0 \end{array} \right\}. \tag{3.7}$$

3.3. **Approximation of the scale functions via finite sum.** The scale function obtained in Proposition 3.1 is an countable sum of exponential functions and in reality its exact value cannot be computed. Here, we obtain bounds for $W^{(q)}(\cdot)$, $W^{(q)\prime}(\cdot)$ and $Z^{(q)}(\cdot)$ in terms of finite sums.

For every $m \geq 1$, let

$$A_{i,q}^{(m)} := 1_{\{i \leq m\}} \left(1 - \frac{\xi_{i,q}}{\eta_i}\right) \prod_{1 \leq j \leq m, j \neq i} \frac{1 - \frac{\xi_{i,q}}{\eta_j}}{1 - \frac{\xi_{i,q}}{\xi_{j,q}}} \quad \text{and} \quad C_{i,q}^{(m)} := \frac{\zeta_q}{q}\frac{\xi_{i,q}A_{i,q}^{(m)}}{\zeta_q + \xi_{i,q}}, \quad i \geq 1.$$

By the interlacing condition, $A_{i,q}^{(m)}$ and $C_{i,q}^{(m)}$ are all positive and, for every $i \geq 1$,

$$A_{i,q}^{(m)} \uparrow A_{i,q} \quad \text{and} \quad C_{i,q}^{(m)} \uparrow C_{i,q} \text{ as } m \to \infty.$$

Now we define candidates for the upper and lower bounds of W_{ζ_q} respectively by

$$\overline{W}_{\zeta_q}^{(m)}(x) := (\psi'(\zeta_q))^{-1} - \sum_{i=1}^{m} C_{i,q}^{(m)} e^{-(\zeta_q + \xi_{i,q})x},$$
$$\underline{W}_{\zeta_q}^{(m)}(x) := \overline{W}_{\zeta_q}^{(m)}(x) - \delta_m \left[e^{-\zeta_q x} + e^{-(\zeta_q + \xi_{m+1,q})x} \right],$$

for every $m \geq 1$ and $x \geq 0$, where

$$\delta_m := \gamma_q - \sum_{i=1}^{m} C_{i,q}^{(m)} > 0,$$

which vanishes in the limit as $m \to \infty$ by (3.4). As candidates for the upper and lower bounds of $W^{(q)}$, we also define

$$\overline{W}^{(q,m)}(x) := e^{\zeta_q x}\overline{W}_{\zeta_q}^{(m)}(x) \quad \text{and} \quad \underline{W}^{(q,m)}(x) := e^{\zeta_q x}\underline{W}_{\zeta_q}^{(m)}(x), \quad x \geq 0.$$

The following proposition shows that the scale functions are bounded and approximated by these functions.

Proposition 3.2. *For every $m \geq 1$ and $x \geq 0$, we have*

$$\underline{W}_{\zeta_q}^{(m)}(x) \leq W_{\zeta_q}(x) \leq \overline{W}_{\zeta_q}^{(m)}(x) \quad \text{and} \quad \underline{W}^{(q,m)}(x) \leq W^{(q)}(x) \leq \overline{W}^{(q,m)}(x). \tag{3.8}$$

Furthermore, we have

$$\overline{W}_{\zeta_q}^{(m)}(x) \xrightarrow{m\uparrow\infty} W_{\zeta_q}(x) \quad \text{and} \quad \overline{W}^{(q,m)}(x) \xrightarrow{m\uparrow\infty} W^{(q)}(x),$$
$$\underline{W}_{\zeta_q}^{(m)}(x) \xrightarrow{m\uparrow\infty} W_{\zeta_q}(x) \quad \text{and} \quad \underline{W}^{(q,m)}(x) \xrightarrow{m\uparrow\infty} W^{(q)}(x),$$

uniformly on $x \in [0, \infty)$.

By straightforward calculation, we can bound $Z^{(q)}$ in (2.6). Let, for every $m \geq 1$,

$$\overline{Z}^{(q,m)}(x) := 1 + q\int_0^x \overline{W}^{(m,q)}(y)\mathrm{d}y \quad \text{and} \quad \underline{Z}^{(q,m)}(x) := 1 + q\int_0^x \underline{W}^{(m,q)}(y)\mathrm{d}y, \quad x \geq 0.$$

Then by Proposition 3.2, we have $\underline{Z}^{(q,m)}(x) \leq Z^{(q,m)}(x) \leq \overline{Z}^{(q,m)}(x)$ and

$$\begin{aligned} 0 \leq \overline{Z}^{(q,m)}(x) - \underline{Z}^{(q,m)}(x) &= q\int_0^x \left(\overline{W}^{(q,m)}(y) - \underline{W}^{(q,m)}(y)\right)\mathrm{d}y \\ &= q\delta_m \int_0^x \left[1 + e^{-\xi_{m+1,q}y}\right]\mathrm{d}y = q\delta_m\left[x + \frac{1}{\xi_{m+1,q}}(1 - e^{-\xi_{m+1,q}x})\right]. \end{aligned}$$

We therefore have the following.

Corollary 3.1 (Bounds on $Z^{(q)}$). *We have $\overline{Z}^{(q,m)}(x) \to Z^{(q)}(x)$ and $\underline{Z}^{(q,m)}(x) \to Z^{(q)}(x)$ as $m \to \infty$ pointwise for every $x \geq 0$.*

We now obtain bounds for the derivative. Define, for every $x > 0$,

$$\underline{w}^{(m)}(x) := (\psi'(\zeta_q))^{-1}\zeta_q e^{\zeta_q x} + \sum_{i=1}^m C_{i,q}^{(m)}\xi_{i,q}e^{-\xi_{i,q}x},$$
$$\overline{w}^{(m)}(x) := \underline{w}^{(m)}(x) + \left[\max_{1\leq k\leq m}(\xi_{k,q}e^{-\xi_{k,q}x}) + \max_{k\geq m+1}(\xi_{k,q}e^{-\xi_{k,q}x})\right]\delta_m.$$

Here notice that

$$\max_{k\geq m+1}(\xi_{k,q}e^{-\xi_{k,q}x}) = \begin{cases} \frac{1}{ex}, & \xi_{m+1,q} \leq \frac{1}{x}, \\ \xi_{m+1,q}e^{-\xi_{m+1,q}x}, & \xi_{m+1,q} > \frac{1}{x}. \end{cases}$$

Proposition 3.3. *For every $m \geq 1$, we have*

$$\underline{w}^{(m)}(x) \leq W^{(q)\prime}(x) \leq \overline{w}^{(m)}(x), \quad x > 0.$$

Furthermore, we have $\underline{w}^{(m)}(x) \to W^{(q)\prime}(x)$ and $\overline{w}^{(m)}(x) \to W^{(q)\prime}(x)$ uniformly on $x \geq x_0$ for any $x_0 > 0$.

A stronger result holds when $\sigma > 0$ or $\Pi(-\infty, 0) < \infty$. Recall in this case that $\theta < \infty$ by Lemma 3.2-(2) and hence we can define

$$\epsilon_m := \theta - \frac{\zeta_q}{q}\sum_{i=1}^m \xi_{i,q}A_{i,q}^{(m)} > 0, \quad m \geq 1,$$

which vanishes in the limit as $m \to \infty$ by Lemma 3.2-(2).

Corollary 3.2. *When $\sigma > 0$ or $\Pi(-\infty, 0) < \infty$, we have*

$$\underline{w}^{(m)}(x) \leq W^{(q)\prime}(x) \leq \overline{w}^{(m)}(x) \wedge \widetilde{w}^{(m)}(x), \quad x > 0$$

where

$$\widetilde{w}^{(m)}(x) := \underline{w}^{(m)}(x) + \max_{1 \leq k \leq m} (\xi_{k,q} e^{-\xi_{k,q} x}) \delta_m + e^{-\xi_{m+1,q} x} \epsilon_m.$$

j	p_j	η_j
1	0.029931	676.178
2	0.093283	38.7090
3	0.332195	4.27400
4	0.476233	0.76100
5	0.068340	0.24800
6	0.000018	0.09700

(i) Weibull(0.6,0.665)

j	p_j	η_j	j	p_j	η_j
1	8.37E-11	8.3E-09	8	0.000147	0.0020
2	7.18E-10	6.8E-08	9	0.001122	0.0100
3	5.56E-09	3.9E-07	10	0.008462	0.0570
4	4.27E-08	2.2E-06	11	0.059768	0.3060
5	3.27E-07	1.2E-05	12	0.307218	1.5460
6	2.50E-06	6.5E-05	13	0.533823	6.5160
7	1.92E-05	3.5E-04	14	0.089437	23.304

(ii) Pareto(1.2,5)

TABLE 1. Parameters of the hyperexponential distributions fitted to (i) Weibull$(0.6, 0.665)$ and (ii) Pareto$(1.2, 5)$ (taken from Tables 3 and 9, respectively, of [15]).

4. Numerical Examples

In this section, we illustrate numerically the effectiveness of the PH-fitting approximation for a general spectrally negative Lévy process. First, we use the classical hyperexponential-fitting algorithm for a completely monotone density function by [15] to approximate the scale function for the case with a (Brownian motion plus) compound Poisson process with Weibull/Pareto-distributed jumps. We approximate (2.5) and (2.24) and compare them with simulated results. Second, we consider, as an example of the meromorphic Lévy process, the β-family introduced by [20] and extend the results to the spectrally negative version of the CGMY process.

4.1. **Numerical results on hyperexponential-fitting.**

As noted earlier, any spectrally negative Lévy process with a completely monotone Lévy density can be approximated arbitrarily closely by hyperexponential-fitting. Here, we use the fitted data computed by [15] to approximate the scale function for a compound Poisson process with i.i.d. Weibull/Pareto-distributed jumps (with or without a Brownian motion component). Recall that the Weibull distribution with parameters c and a (denoted Weibull(c, a)) is give by

$$F(t) = 1 - e^{-(t/a)^c}, \quad t \geq 0$$

and the Pareto distribution with positive parameters a and b (denoted Pareto(a, b)) is given by

$$F(t) = 1 - (1 + bt)^{-a}, \quad t \geq 0.$$

These have long-tails, namely, $e^{\delta t}(1 - F(t)) \to \infty$ as $t \uparrow \infty$ for any $\delta > 0$. See [19] for more details about these distributions. Feldman and Whitt [15] constructed a recursive algorithm to approximate completely monotone

	$\sigma = 1$		$\sigma = 0$	
x	scale function	simulation	scale function	simulation
1	.30312	.30256 (.30013, .30498)	.30977	.31053 (.30797, .31309)
2	.46728	.46793 (.46517, .47068)	.47102	.47113 (.46792, .47435)
3	.63707	.63642 (.63370, .63914)	.63862	.63662 (.63372, .63952)
4	.81409	.81231 (.81023, .81439)	.81433	.81546 (.81310, .81782)

Exp(1)

	$\sigma = 1$		$\sigma = 0$	
x	scale function	simulation	scale function	simulation
1	.40683	.40420 (.40116, .40724)	.41734	.41517 (.41225, .41809)
2	.55851	.55473 (.55176, .55769)	.56526	.56394 (.56080, .56707)
3	.70570	.70013 (.69767, .70259)	.70968	.70491 (.70229, .70753)
4	.85210	.84925 (.84740, .85111)	.85385	.85240 (.85017, .85462)

Weibull(0,6,0.665)

	$\sigma = 1$		$\sigma = 0$	
x	scale function	simulation	scale function	simulation
1	.69953	.70291 (.70042, .70540)	.72042	.72287 (.72041, .72533)
2	.80667	.80782 (.80569, .80994)	.81614	.81729 (.81513, .81945)
3	.88181	.88176 (.88003, .88348)	.88631	.88663 (.88491, .88835)
4	.94412	.94364 (.94248, .94480)	.94590	.94747 (.94635, .94859)

Pareto(1.2,5)

TABLE 2. Computation of $\mathbb{E}^x[e^{-q\tau_b^+} 1_{\{\tau_0^- > \tau_b^+, \tau_b^+ < \infty\}}]$ via scale function and simulation. The simulation column shows the mean and the 95% confidence interval.

densities in terms of hyperexponential densities. We use their results and approximate the scale functions of spectrally negative Lévy processes with Weibull/Pareto-distributed jumps.

We consider the Lévy processes $X^{(\text{weibull})}$ and $X^{(\text{pareto})}$ in the form (2.12) where Z is (i) Weibull(0.6,0.665) and (ii) Pareto(1.2,5), respectively. Table 1 shows the parameters of the hyperexponential distributions obtained by [15] fitted to (i) with $m = 6$ and to (ii) with $m = 14$. We use these parameters to construct hyperexponential Lévy processes $\widetilde{X}^{(\text{weibull})}$ and $\widetilde{X}^{(\text{pareto})}$ (see Example 2.1) that will be used to approximate $X^{(\text{weibull})}$ and $X^{(\text{pareto})}$, respectively.

First, we study how accurately the scale function of $X^{(\text{weibull})}$ (resp. $X^{(\text{pareto})}$) can be approximated by those of $\widetilde{X}^{(\text{weibull})}$ ($\widetilde{X}^{(\text{pareto})}$). Toward this end, we use the identity as in (2.5)

$$\mathbb{E}^x \left[e^{-q\tau_b^+} 1_{\{\tau_0^- > \tau_b^+, \tau_b^+ < \infty\}} \right] = W^{(q)}(x)/W^{(q)}(b). \tag{4.1}$$

We compute the right-hand side for $\widetilde{X}^{(\text{weibull})}$ ($\widetilde{X}^{(\text{pareto})}$) explicitly via (2.20) and approximate the left-hand side for $X^{(\text{weibull})}$ ($X^{(\text{pareto})}$) via Monte Carlo simulation based on $100,000$ sample paths. For the simulated results, Brownian motions are approximated by random walks with time step $\Delta t = T/100$ for each interarrival time T

between jumps. In order to confirm the accuracy of the simulated results, we also calculate both sides for the process $X^{(\text{exp})}$ with i.i.d. exponential jumps with parameter 1. For the scale functions of $X^{(\text{exp})}$, $\widetilde{X}^{(\text{weibull})}$ and $\widetilde{X}^{(\text{pareto})}$, the roots $\xi_{\cdot,q}$'s and ζ_q are obtained via the bisection method with error bound $1.0E-10$.

We consider the case $\sigma = 0$ and $\sigma = 1$ and various values of starting point x with common parameters $q = 0.05$, $\lambda = 5$, $b = 5$ and $\mu = 5$. Table 2 gives the results. From these, we see that (4.1) for $X^{(\text{weibull})}$ and $X^{(\text{pareto})}$ are approximated very precisely. Based on the accuracy for each $x = 1, \ldots, 4$, we can also infer that their scale functions are approximated efficiently. Their scale functions and first derivatives for $X^{(\text{exp})}$, $\widetilde{X}^{(\text{weibull})}$ and $\widetilde{X}^{(\text{pareto})}$ are plotted in Figure 1. It can be easily confirmed that the results are consistent with Lemma 2.2; in particular, $W^{(q)}(0) = 1/\mu = 0.2$ and $W^{(q)\prime}(0+) = (q+\lambda)/(\mu^2) = 0.202$ for $\sigma = 0$ whereas $W^{(q)}(0) = 0$ and $W^{(q)\prime}(0+) = 2/(\sigma^2) = 2$ for $\sigma = 1$.

Second, we evaluate the approximation of (2.24) for $X^{(\text{weibull})}$ ($X^{(\text{pareto})}$) using those for $\widetilde{X}^{(\text{weibull})}$ ($\widetilde{X}^{(\text{pareto})}$). Here we compute the overshoot/undershoot density

$$\mathbb{E}^x\left[e^{-q\tau_0^-} 1_{\{-X_{\tau_0^-} \in \mathrm{d}a,\, \tau_0^- < \infty\}}\right] \quad \text{and} \quad \mathbb{E}^x\left[e^{-q\tau_0^-} 1_{\{X_{\tau_0^- -} \in \mathrm{d}b,\, \tau_0^- < \infty\}}\right].$$

These values are computed explicitly by (2.25) for $\widetilde{X}^{(\text{weibull})}$ ($\widetilde{X}^{(\text{pareto})}$). For $X^{(\text{weibull})}$ ($X^{(\text{pareto})}$), we simulate

$$\mathbb{E}^x\left[e^{-q\tau_0^-} 1_{\{-X_{\tau_0^-} \in (a-\Delta a/2, a+\Delta a/2),\, \tau_0^- < \infty\}}\right]/\Delta a \quad \text{and} \quad \mathbb{E}^x\left[e^{-q\tau_0^-} 1_{\{X_{\tau_0^- -} \in (b-\Delta b/2, b+\Delta b/2),\, \tau_0^- < \infty\}}\right]/\Delta b$$

with $\Delta a = \Delta b = 0.1$ by Monte Carlo simulation with $500{,}000$ samples. These values are also computed for $X^{(\text{exp})}$.

Figures 2-3 show the results for the cases $\sigma = 1$ and $\sigma = 0$ with common parameters $x = 5$, $\mu = 1$, $\lambda = 10$ and $q = 0.05$. In Figure 3, the density has a jump at the initial position $x = 5$ for the case $\sigma = 0$ (while it is continuous for the case $\sigma = 1$) due to the fact that $x = 5$ is irregular for $(0, 5)$. As can be seen from these figures, the approximation accurately captures the overshoot/undershoot density for $X^{(\text{weibull})}$ and $X^{(\text{pareto})}$. The spike at the initial position in Figure 3 is precisely realized thanks to the closed-form expression (2.25); this would be difficult to realize if the scale function is approximated via numerical Laplace inversion.

4.2. **Numerical results on the β-class and CGMY process.** We now consider, as an example of meromorphic Lévy processes, the β-class introduced by [20]. The following definition is due to [20], Definition 4.

Definition 4.1. *A spectrally negative Lévy process is said to be in the β-class if its Lévy density is in the form*

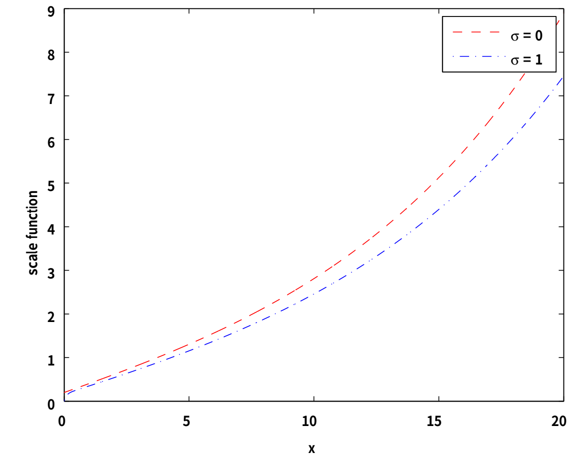
$$\pi(x) = c\frac{e^{\alpha\beta x}}{(1-e^{\beta x})^{\lambda}} 1_{\{x<0\}}, \quad x \in \mathbb{R}, \tag{4.2}$$

for some $\alpha > 0$, $\beta > 0$, $c \geq 0$ and $\lambda \in (0, 3)$. It is equivalent to saying that its Laplace exponent is

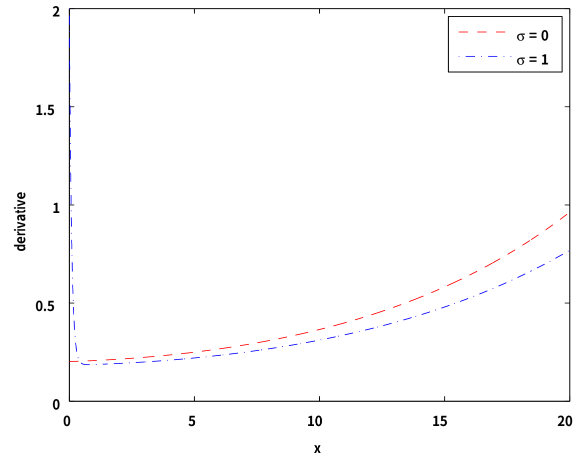
$$\psi(z) = \hat{\mu} z + \frac{1}{2}\sigma^2 z^2 + \frac{c}{\beta}\left\{B(\alpha + \frac{z}{\beta}, 1-\lambda) - B(\alpha, 1-\lambda)\right\}$$

where B is the beta function, i.e., $B(x,y) := \Gamma(x)\Gamma(y)/\Gamma(x+y)$.

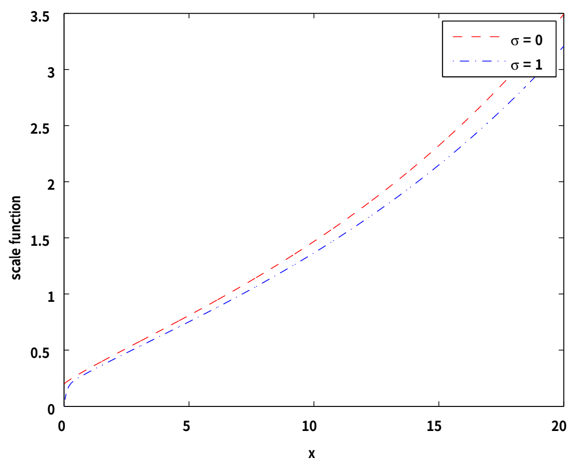
The special case $\sigma = 0$ and $\beta = 1$ reduces to the class of *Lamperti-stable* processes, which are obtained by the Lamperti transformation [31] from the stable processes conditioned to stay positive; see [8, 9] and references therein. For the scale function of a related process, see [28].



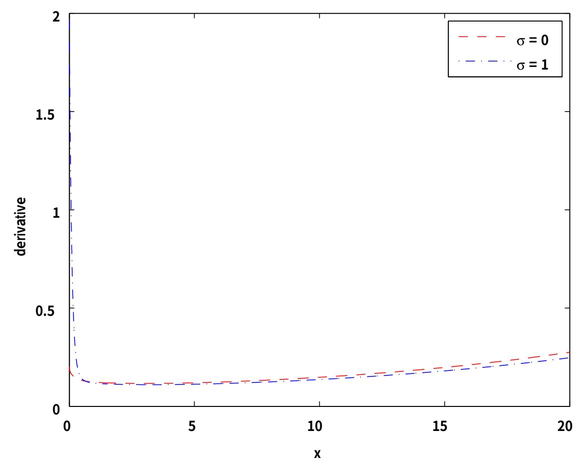
Scale function for the Exp(1) case



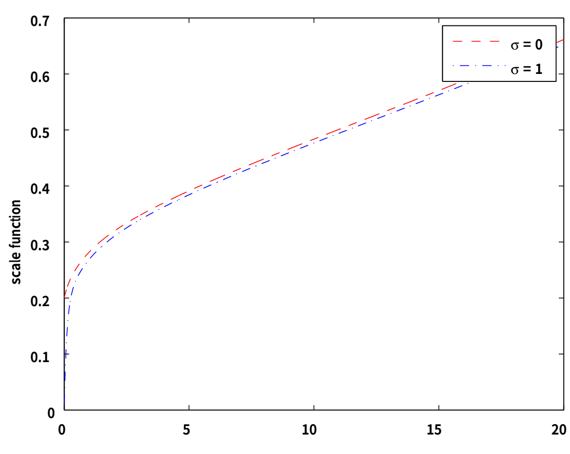
Derivative for the Exp(1) case



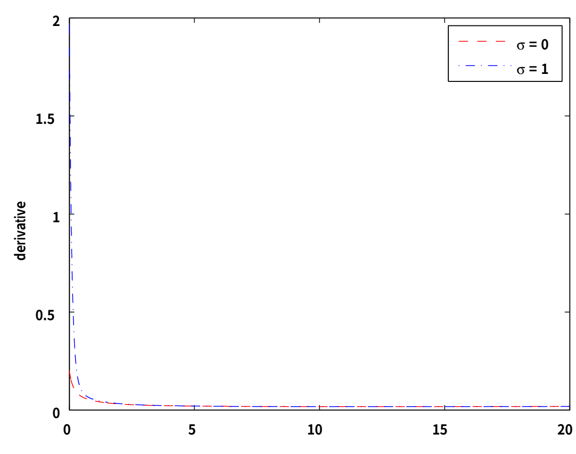
Scale function for the Weibull(0.6,0.665) case



Derivative for the Weibull(0.6,0.665) case

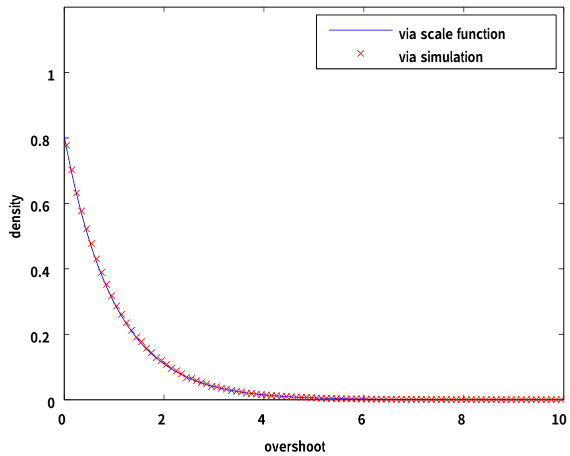


Scale function for the Pareto(1.2,5) case

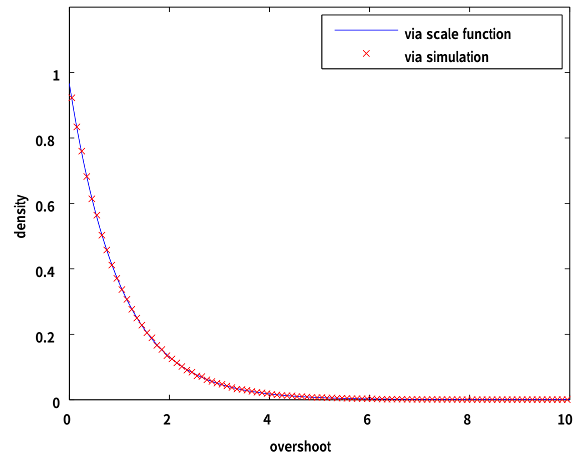


Derivative for the Pareto(1.2,5) case

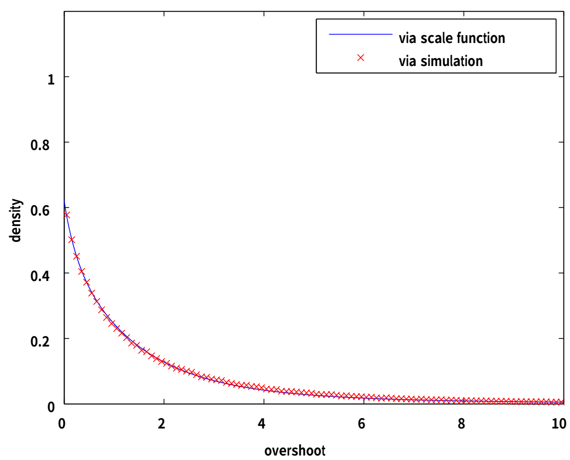
FIGURE 1. Scale functions and their derivatives.



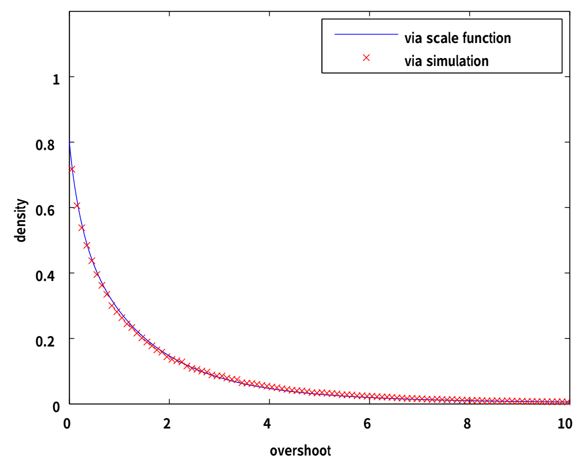
Exp(1) with $\sigma = 1$



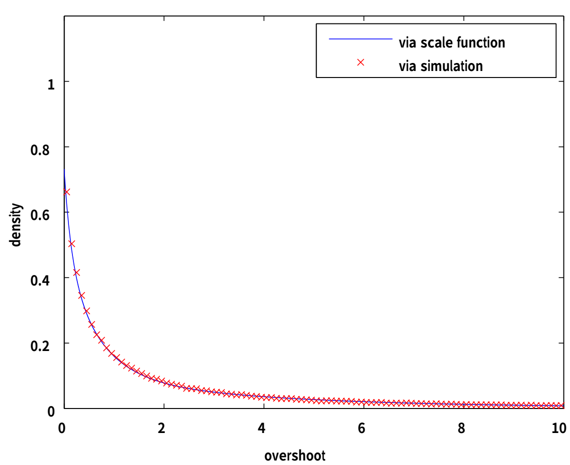
Exp(1) with $\sigma = 0$



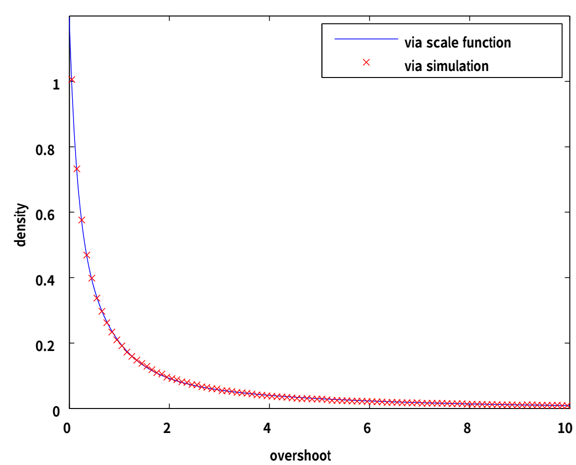
Weibull(0.6,0.665) with $\sigma = 1$



Weibull(0.6,0.665) with $\sigma = 0$

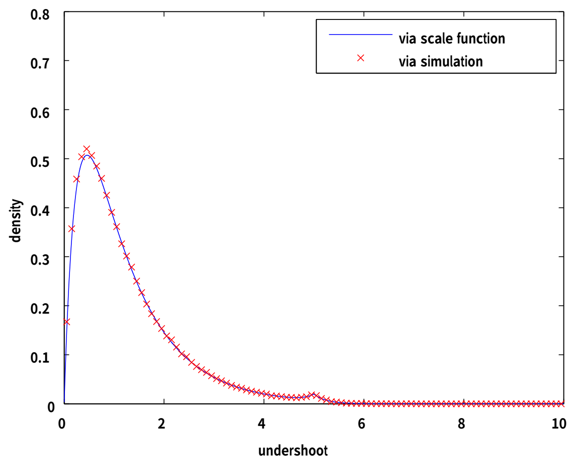


Pareto(1.2,5) with $\sigma = 1$

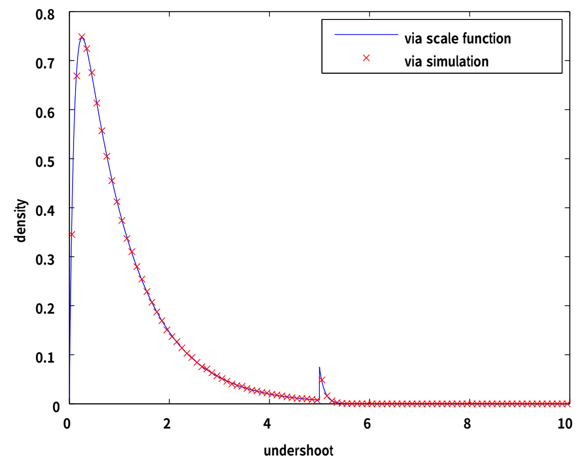


Pareto(1.2,5) with $\sigma = 0$

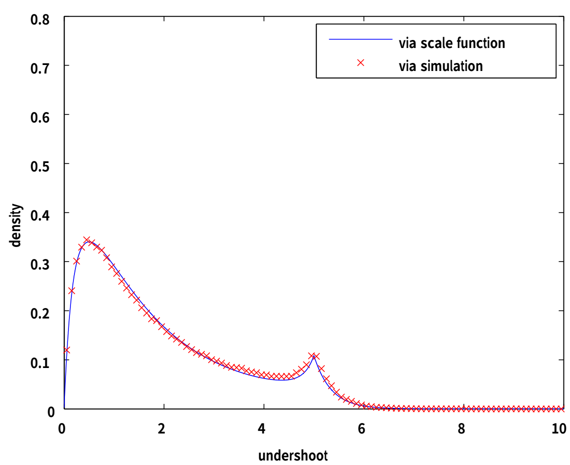
FIGURE 2. Computation of the overshoot density $\mathbb{E}^x[e^{-q\tau_0^-} 1_{\{-X_{\tau_0^-}\in \mathrm{d}a, \tau_0^-<\infty\}}]$.



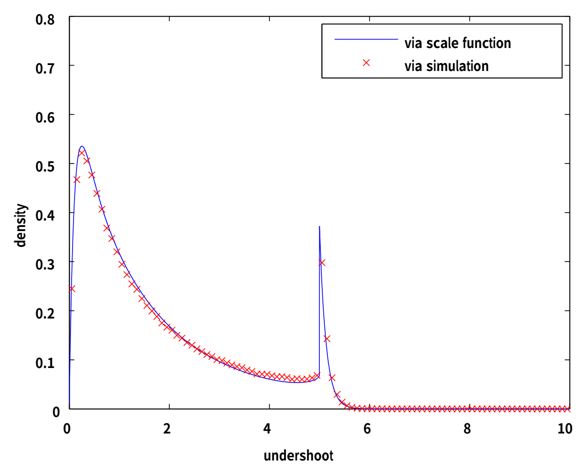
Exp(1) with $\sigma = 1$



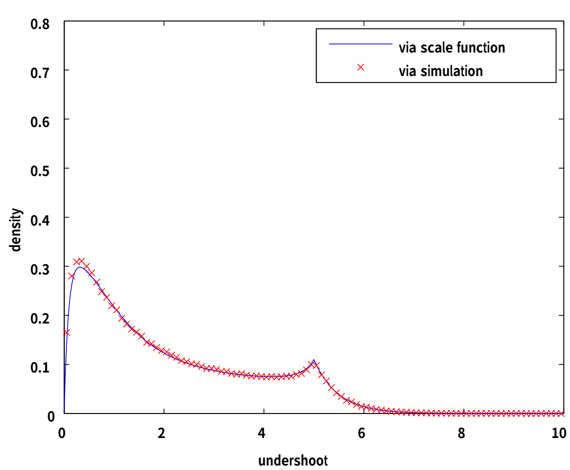
Exp(1) with $\sigma = 0$



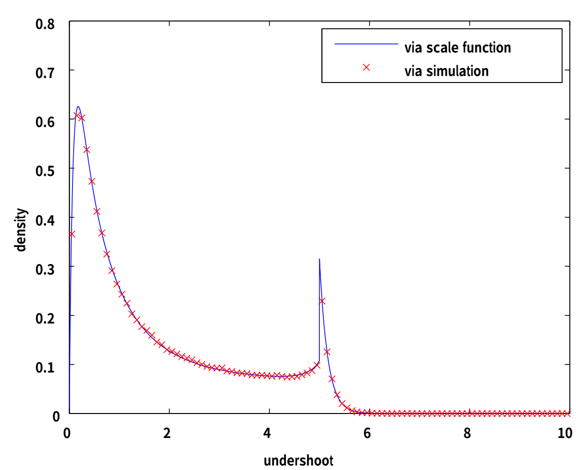
Weibull(0.6,0.665) with $\sigma = 1$



Weibull(0.6,0.665) with $\sigma = 0$



Pareto(1.2,5) with $\sigma = 1$



Pareto(1.2,5) with $\sigma = 0$

FIGURE 3. Computation of the undershoot density $\mathbb{E}^x[e^{-q\tau_0^-} 1_{\{X_{\tau_0^- -} \in \mathrm{d}b, \tau_0^- < \infty\}}]$.

It can be also seen that this is a "discrete-version" of the (spectrally negative) *CGMY* process, whose Lévy density is given by

$$\pi(x) = c\frac{e^{\alpha x}}{|x|^{\lambda}} 1_{\{x<0\}}, \quad x \in \mathbb{R}. \tag{4.3}$$

Indeed, if we set $c = \widetilde{c}\beta^{\lambda}$ and $\alpha = \widetilde{\alpha}\beta^{-1}$ in (4.2), we have

$$c\frac{e^{\alpha\beta x}}{(1-e^{\beta x})^{\lambda}} 1_{\{x<0\}} \xrightarrow{\beta \downarrow 0} \widetilde{c}\frac{e^{\widetilde{\alpha} x}}{|x|^{\lambda}} 1_{\{x<0\}}, \quad x \in \mathbb{R}. \tag{4.4}$$

See [4] for approximation of (double-sided) CGMY processes using hyperexponential distributions.

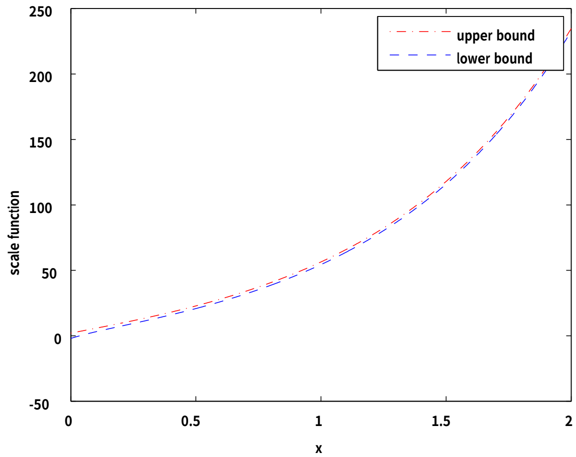
We use the results in Section 3 to obtain the bounds on the scale functions. We consider the case $q = 0.03$, $\sigma = 0.2$, $\hat{\mu} = 0.1$, $\lambda = 1.5$, $\alpha = 3$, $\beta = 1$ and $c = 0.1$ in (4.2). It is easy to verify that these have jumps of infinite activity (and bounded variation). First, we plot, in Figure 4, the upper and lower bounds on the scale function and its derivative as obtain in Proposition 3.2 and Corollary 3.2 for $m = 10$ and $m = 100$. The difference between the upper and lower bounds indeed converges to zero, although the convergence of the derivative in the neighborhood of zero is relatively slow. Second, motivated by (4.4), we take β to zero and see how the approximation for the CGMY process works. Here we set $\tilde{\alpha} = 3$, $\tilde{c} = 0.1$ and $m = 100$ and use the same values as the above for the other parameters. Figure 5 shows the upper and lower bounds of scale function and its derivative for various values of β; we can indeed observe the convergence as $\beta \to 0$.

5. Concluding Remarks

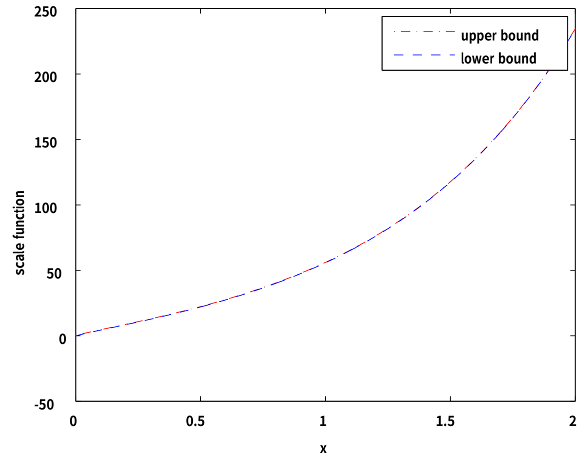
We have studied the scale function for the spectrally negative PH Lévy process and the PH-fitting approach for the approximation of the scale function for a general spectrally negative Lévy process. Because the approximated scale function is given as a function in a closed form, one can analytically differentiate/integrate to obtain other fluctuation identities explicitly. Our numerical results suggest that the PH-fitting is a powerful alternative to the numerical Laplace inversion approach, particularly when the Lévy density is completely monotone.

One major challenge for the PH-fitting approach is that, without the completely monotone assumption, there does not exist a fitting algorithm that is guaranteed to work for arbitrary measure. The performance is certainly dependent on the shape of the Lévy measure and it may not be a suitable approach for certain cases. However, there exist a variety of fitting algorithms typically developed in queueing analysis. Well-known examples are the moment-matching approach (e.g. MEFIT and MEDA) and the maximum-likelihood approach (e.g. MLAPH and EMPHT), and a thorough study of pros and cons of each fitting techniques has been conducted in, for example, [17, 32]. Our next step is, therefore, to apply these existing algorithms for the approximation of the scale function, and analyze its performance for a variety of Lévy measures. Another direction for future research is the PH-fitting construction of scale functions from empirical data as in, for example, [4]. The closed-form expression of the approximated scale function can be used flexibly to identify the fluctuation of the process implied by the empirical data.

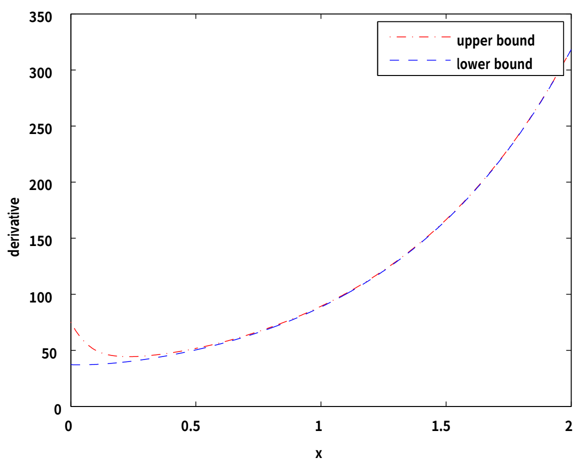
Appendix A. Proofs



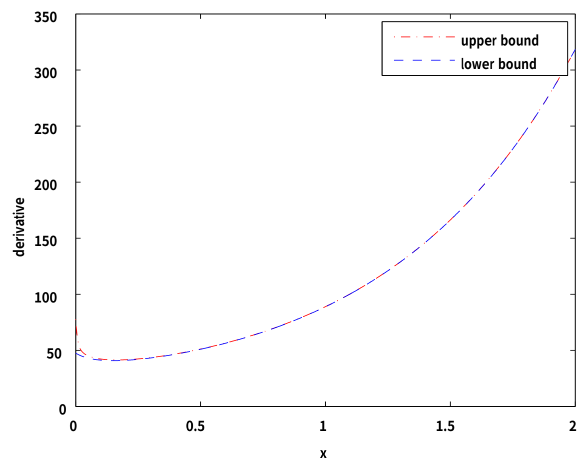
Bounds on $W^{(q)}(x)$ when $m = 10$



Bounds on $W^{(q)}(x)$ when $m = 100$



Bounds on $W^{(q)\prime}(x)$ when $m = 10$

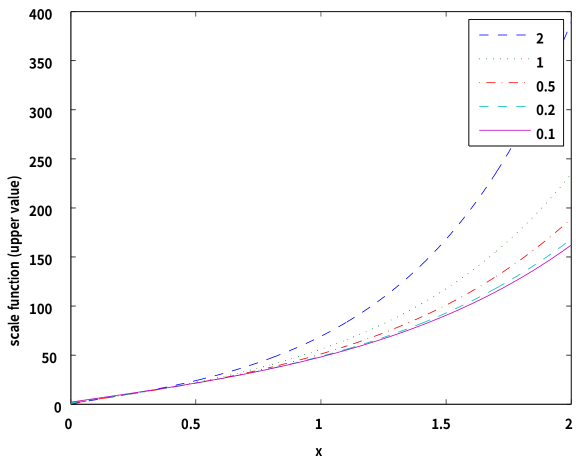


Bounds on $W^{(q)\prime}(x)$ when $m = 100$

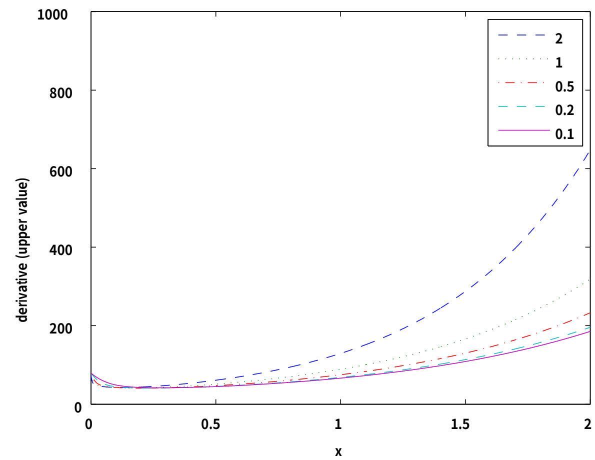
FIGURE 4. Approximation of the scale function and its derivative for the β-class.

A.1. **Proof of Lemmas 2.3-2.4.** By (2.17), it is easy to verify that

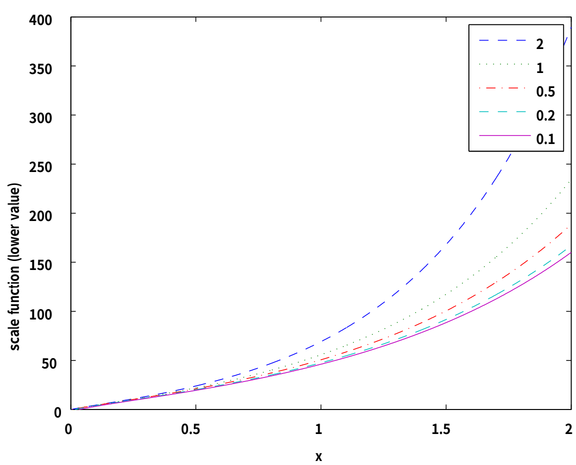
$$\mathbb{E}^x \left[e^{-q\tau_a^-} 1_{\{\tau_a^- < \infty\}} \right] = \sum_{i=1}^{n} \sum_{k=1}^{m_i} A_{i,q}^{(k)} \xi_{i,q} \int_{x-a}^{\infty} \frac{(\xi_{i,q} y)^{k-1}}{(k-1)!} e^{-\xi_{i,q} y} \mathrm{d}y, \quad 0 \le a < x,$$



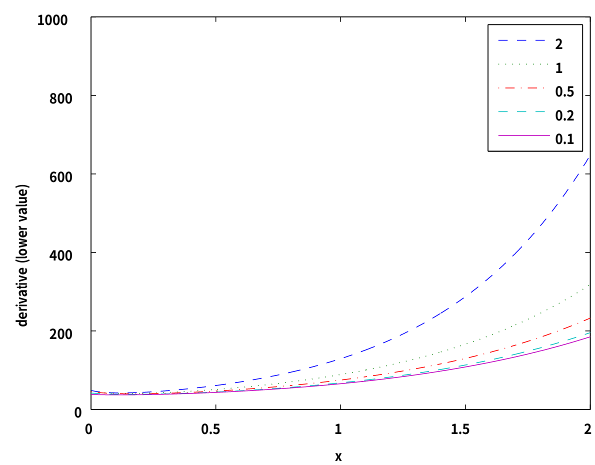
upper bounds for the scale functions



upper bounds for the derivatives



lower bounds for the scale functions



lower bounds for the derivatives

FIGURE 5. Convergence to the CGMY model.

and hence, because $W^{(q)} \in C^1(0,\infty)$,

$$\frac{\partial}{\partial a}\mathbb{E}^x\left[e^{-q\tau_a^-}1_{\{\tau_a^-<\infty\}}\right] = \sum_{i=1}^n \sum_{k=1}^{m_i} A_{i,q}^{(k)} \xi_{i,q} \frac{(\xi_{i,q}(x-a))^{k-1}}{(k-1)!} e^{-\xi_{i,q}(x-a)}, \quad 0 \leq a < x, \tag{A.1}$$

$$\frac{\partial}{\partial x}\mathbb{E}^x\left[e^{-q\tau_0^-}1_{\{\tau_0^-<\infty\}}\right]\bigg|_{x=0+} = -\sum_{i=1}^n A_{i,q}^{(1)}\xi_{i,q}. \tag{A.2}$$

On the other hand, different representations of (A.1)-(A.2) can be pursued. By Theorem 8.1 of [25] and (2.7),

$$\mathbb{E}^x\left[e^{-q\tau_a^-}1_{\{\tau_a^-<\infty\}}\right] = Z^{(q)}(x-a) - \frac{q}{\zeta_q}W^{(q)}(x-a) = 1 + q\int_0^{x-a} W^{(q)}(y)\mathrm{d}y - \frac{q}{\zeta_q}e^{\zeta_q(x-a)}W_{\zeta_q}(x-a)$$

for every $0 \leq a < x$. Its derivative with respect to a becomes

$$\begin{aligned}\frac{\partial}{\partial a}\mathbb{E}^x\left[e^{-q\tau_a^-}1_{\{\tau_a^-<\infty\}}\right] &= -qW^{(q)}(x-a) + qe^{\zeta_q(x-a)}W_{\zeta_q}(x-a) + \frac{q}{\zeta_q}e^{\zeta_q(x-a)}W'_{\zeta_q}(x-a) \\ &= \frac{q}{\zeta_q}e^{\zeta_q(x-a)}W'_{\zeta_q}(x-a).\end{aligned} \tag{A.3}$$

In particular, when $a = 0$, the derivative with respect to x and its limit as $x \to 0$ are

$$\frac{\partial}{\partial x}\mathbb{E}^x\left[e^{-q\tau_0^-}1_{\{\tau_0^-<\infty\}}\right] = -\frac{q}{\zeta_q}\left[-\zeta_q W^{(q)}(x) + W^{(q)\prime}(x)\right] \xrightarrow{x\downarrow 0+} = -\frac{q}{\zeta_q}\theta. \tag{A.4}$$

By matching (A.2) and (A.4), Lemma 2.3 is immediate.

For the proof of Lemma 2.4, by matching (A.1) and (A.3), we have

$$\begin{aligned}W'_{\zeta_q}(y) &= \frac{\zeta_q}{q}\sum_{i=1}^n\sum_{k=1}^{m_i}A_{i,q}^{(k)}\xi_{i,q}\frac{(\xi_{i,q}y)^{k-1}}{(k-1)!}e^{-(\zeta_q+\xi_{i,q})y} \\ &= \frac{\zeta_q}{q}\sum_{i=1}^n\sum_{k=1}^{m_i}A_{i,q}^{(k)}(\zeta_q+\xi_{i,q})\left(\frac{\xi_{i,q}}{\zeta_q+\xi_{i,q}}\right)^k\frac{((\zeta_q+\xi_{i,q})y)^{k-1}}{(k-1)!}e^{-(\zeta_q+\xi_{i,q})y}, \quad y \geq 0.\end{aligned}$$

Integrating the above and changing variables, we have

$$W_{\zeta_q}(x) - W_{\zeta_q}(0) = \frac{\zeta_q}{q}\sum_{i=1}^n\sum_{k=1}^{m_i}A_{i,q}^{(k)}\left(\frac{\xi_{i,q}}{\zeta_q+\xi_{i,q}}\right)^k\frac{1}{(k-1)!}\int_0^{(\zeta_q+\xi_{i,q})x}z^{k-1}e^{-z}\mathrm{d}z, \quad x \geq 0.$$

Lemma 2.4 is now immediate because the integral part is a lower incomplete gamma function.

A.2. **Proof of Lemma 2.5.** Let $N(\cdot,\cdot)$ be the Poisson random measure for $-X$ and $\underline{X}_t := \min_{0\leq s\leq t}X_s$ for all $t \geq 0$. By the compensation formula, we have

$$\begin{aligned}\mathbb{E}^x\left[e^{-q\tau_0^-}1_{\{X_{\tau_0^- -}\in B,\, X_{\tau_0^-}\in A\}}\right] &= \mathbb{E}^x\left[\int_0^\infty\int_0^\infty N(\mathrm{d}t,\mathrm{d}u)e^{-qt}1_{\{X_{t-}-u\in A,\, X_{t-}\in B,\, \underline{X}_{t-}>0\}}\right] \\ &= \mathbb{E}^x\left[\int_0^\infty e^{-qt}\mathrm{d}t\int_0^\infty \overline{\Pi}(\mathrm{d}u)1_{\{X_{t-}-u\in A,\, X_{t-}\in B,\, \underline{X}_{t-}>0\}}\right] \\ &= \int_0^\infty \overline{\Pi}(\mathrm{d}u)\int_0^\infty \mathrm{d}t\left[e^{-qt}\mathbb{P}^x\{X_{t-}\in B\cap(A+u),\, \underline{X}_{t-}>0\}\right] \\ &= \int_0^\infty \overline{\Pi}(\mathrm{d}u)\int_0^\infty \mathrm{d}t\left[e^{-qt}\mathbb{P}^x\{X_{t-}\in B\cap(A+u),\, \tau_0^-\geq t\}\right].\end{aligned}$$

By using the q-resolvent kernel (see, e.g., [25])

$$\begin{aligned}\int_0^\infty \mathrm{d}t\left[e^{-qt}\mathbb{P}^x\{X_{t-}\in B\cap(A+u), \tau_0^-\geq t\}\right] &= \int_{B\cap(A+u)}\mathrm{d}y\left[e^{-\zeta_q y}W^{(q)}(x) - W^{(q)}(x-y)\right] \\ &= W^{(q)}(x)\int_{B\cap(A+u)}e^{-\zeta_q y}\mathrm{d}y - \int_{B\cap(A+u)}\mathrm{d}yW^{(q)}(x-y).\end{aligned}$$

Substituting this, we obtain the result.

A.3. **Proof of Proposition 2.2.** Define for all $B \in \mathcal{B}(0,\infty)$, $A \in \mathcal{B}(-\infty,0)$ and $K \in \mathbb{R}$

$$\rho(K;A,B) := \int_0^\infty \overline{\Pi}(\mathrm{d}u) \int_{B\cap(A+u)} e^{Ky}\mathrm{d}y.$$

Because by Corollary 2.1 and (2.21)

$$\int_0^\infty \overline{\Pi}(\mathrm{d}u) W^{(q)}(x) \int_{B\cap(A+u)} e^{-\zeta_q y}\mathrm{d}y = W^{(q)}(x)\rho(-\zeta_q;A,B) = \left(\frac{1}{\psi'(\zeta_q)}e^{\zeta_q x} - \sum_{i\in\mathcal{I}_q} C_{i,q}e^{-\xi_{i,q}x}\right)\rho(-\zeta_q;A,B),$$

and

$$\begin{aligned}
&\int_0^\infty \overline{\Pi}(\mathrm{d}u) \int_{B\cap(0,x)\cap(A+u)} W^{(q)}(x-y)\mathrm{d}y \\
&= \int_0^\infty \overline{\Pi}(\mathrm{d}u) \int_{B\cap(0,x)\cap(A+u)} \left(\frac{1}{\psi'(\zeta_q)}e^{\zeta_q(x-y)} - \sum_{i\in\mathcal{I}_q} C_{i,q}e^{-\xi_{i,q}(x-y)}\right)\mathrm{d}y \\
&= \frac{e^{\zeta_q x}}{\psi'(\zeta_q)}\rho(-\zeta_q;A,B\cap(0,x)) - \sum_{i\in\mathcal{I}_q} C_{i,q}e^{-\xi_{i,q}x}\rho(\xi_{i,q};A,B\cap(0,x)),
\end{aligned}$$

we obtain via Lemma 2.5

$$\begin{aligned}
h_q(x;A,B) &= \left(\frac{e^{\zeta_q x}}{\psi'(\zeta_q)} - \sum_{i\in\mathcal{I}_q} C_{i,q}e^{-\xi_{i,q}x}\right)\rho(-\zeta_q;A,B) \\
&\quad - \frac{e^{\zeta_q x}}{\psi'(\zeta_q)}\rho(-\zeta_q;A,B\cap(0,x)) + \sum_{i\in\mathcal{I}_q} C_{i,q}e^{-\xi_{i,q}x}\rho(\xi_{i,q};A,B\cap(0,x)) \\
&= \frac{e^{\zeta_q x}}{\psi'(\zeta_q)}\rho(-\zeta_q;A,B\cap(x,\infty)) + \sum_{i\in\mathcal{I}_q} C_{i,q}e^{-\xi_{i,q}x}\left\{\rho(\xi_{i,q};A,B\cap(0,x)) - \rho(-\zeta_q;A,B)\right\}.
\end{aligned}$$

Lemma A.1. *Suppose $B = (\underline{b},\overline{b})$ and $A = (-\overline{a},-\underline{a})$. For any $K \in \mathbb{R}$, we have*

$$\rho(K;A,B) = \sum_{j=1}^m \frac{\lambda p_j}{\eta_j - K}(e^{-\eta_j\underline{a}} - e^{-\eta_j\overline{a}})\left(e^{-(\eta_j-K)\underline{b}} - e^{-(\eta_j-K)\overline{b}}\right).$$

Proof. For any $u > 0$

$$\begin{aligned}
\int_{B\cap(A+u)} e^{Ky}\mathrm{d}y &= \int_{(\underline{b},\overline{b})\cap(-\overline{a}+u,-\underline{a}+u)} e^{Ky}\mathrm{d}y \\
&= \int_{((-\overline{a}+u)\vee\underline{b},(-\underline{a}+u)\wedge\overline{b})} 1_{\{\underline{a}+\underline{b}<u<\overline{a}+\overline{b}\}}e^{Ky}\mathrm{d}y = \frac{1}{K}\left[e^{K((-\underline{a}+u)\wedge\overline{b})} - e^{K((-\overline{a}+u)\vee\underline{b})}\right]1_{\{\underline{a}+\underline{b}<u<\overline{a}+\overline{b}\}}.
\end{aligned}$$

For its integral with respect to $\overline{\Pi}$,

$$\int_{\underline{a}+\underline{b}}^{\overline{a}+\overline{b}} \overline{\Pi}(\mathrm{d}u)e^{K((-\underline{a}+u)\wedge\overline{b})} = \int_{\underline{a}+\underline{b}}^{\underline{a}+\overline{b}} \overline{\Pi}(\mathrm{d}u)e^{K(-\underline{a}+u)} + \int_{\underline{a}+\overline{b}}^{\overline{a}+\overline{b}} \overline{\Pi}(\mathrm{d}u)e^{K\overline{b}} = e^{K\overline{b}}\int_{\underline{a}}^{\overline{a}} \overline{\Pi}(\mathrm{d}u+\overline{b}) + \int_{\underline{b}}^{\overline{b}} \overline{\Pi}(\mathrm{d}u+\underline{a})e^{Ku},$$

$$\int_{\underline{a}+\underline{b}}^{\overline{a}+\overline{b}} \overline{\Pi}(\mathrm{d}u)e^{K((-\overline{a}+u)\vee\underline{b})} = \int_{\underline{a}+\underline{b}}^{\overline{a}+\underline{b}} \overline{\Pi}(\mathrm{d}u)e^{K\underline{b}} + \int_{\overline{a}+\underline{b}}^{\overline{a}+\overline{b}} \overline{\Pi}(\mathrm{d}u)e^{K(-\overline{a}+u)} = e^{K\underline{b}}\int_{\underline{a}}^{\overline{a}} \overline{\Pi}(\mathrm{d}u+\underline{b}) + \int_{\underline{b}}^{\overline{b}} \overline{\Pi}(\mathrm{d}u+\overline{a})e^{Ku}.$$

Therefore

$$\rho(K;A,B)=\frac{1}{K}\left(e^{K\bar{b}}\int_{\underline{a}}^{\overline{a}}\overline{\Pi}(\mathrm{d}u+\bar{b})+\int_{\underline{b}}^{\bar{b}}\overline{\Pi}(\mathrm{d}u+\underline{a})e^{Ku}-e^{K\underline{b}}\int_{\underline{a}}^{\overline{a}}\overline{\Pi}(\mathrm{d}u+\underline{b})-\int_{\underline{b}}^{\bar{b}}\overline{\Pi}(\mathrm{d}u+\overline{a})e^{Ku}\right).$$

Using $\overline{\Pi}(\mathrm{d}u)=\lambda\sum_{j=1}^{m}p_j\eta_je^{-\eta_ju}\mathrm{d}u$, for any $\underline{\beta}$, $\overline{\beta}$ and α,

$$\int_{\underline{\beta}}^{\overline{\beta}}\overline{\Pi}(\mathrm{d}u+\alpha)e^{Ku}=\lambda\sum_{j=1}^{m}p_j\eta_je^{-\eta_j\alpha}\int_{\underline{\beta}}^{\overline{\beta}}e^{-(\eta_j-K)u}=\lambda\sum_{j=1}^{m}\frac{p_j\eta_j}{\eta_j-K}e^{-\eta_j\alpha}\left(e^{-(\eta_j-K)\underline{\beta}}-e^{-(\eta_j-K)\overline{\beta}}\right).$$

Thus

$$\begin{aligned}\rho(K;A,B)&=\frac{\lambda}{K}\sum_{j=1}^{m}\frac{p_j\eta_j}{\eta_j-K}(e^{-\eta_j\underline{a}}-e^{-\eta_j\overline{a}})\left(e^{-(\eta_j-K)\underline{b}}-e^{-(\eta_j-K)\bar{b}}\right)\\&-\frac{\lambda}{K}\sum_{j=1}^{m}p_j(e^{-(\eta_j-K)\underline{b}}-e^{-(\eta_j-K)\bar{b}})\left(e^{-\eta_j\underline{a}}-e^{-\eta_j\overline{a}}\right)\\&=\lambda\sum_{j=1}^{m}\frac{p_j}{\eta_j-K}(e^{-\eta_j\underline{a}}-e^{-\eta_j\overline{a}})\left(e^{-(\eta_j-K)\underline{b}}-e^{-(\eta_j-K)\bar{b}}\right).\end{aligned}$$

□

We are now ready to prove Proposition 2.2. By Lemma A.1,

$$e^{\zeta_qx}\rho(-\zeta_q;A,B\cap(x,\infty))=e^{\zeta_qx}\lambda\sum_{j=1}^{m}\frac{p_j}{\eta_j+\zeta_q}(e^{-\eta_j\underline{a}}-e^{-\eta_j\overline{a}})\left(e^{-(\eta_j+\zeta_q)(\underline{b}\vee x)}-e^{-(\eta_j+\zeta_q)(\bar{b}\vee x)}\right),$$

$$\sum_{i\in\mathcal{I}_q}C_{i,q}e^{-\xi_{i,q}x}\rho(-\zeta_q;A,B)=\lambda\sum_{i\in\mathcal{I}_q}C_{i,q}e^{-\xi_{i,q}x}\sum_{j=1}^{m}\frac{p_j}{\eta_j+\zeta_q}(e^{-\eta_j\underline{a}}-e^{-\eta_j\overline{a}})\left(e^{-(\eta_j+\zeta_q)\underline{b}}-e^{-(\eta_j+\zeta_q)\bar{b}}\right),$$

$$\begin{aligned}&\sum_{i\in\mathcal{I}_q}C_{i,q}e^{-\xi_{i,q}x}\rho(\xi_{i,q};A,B\cap(0,x))\\&=\lambda\sum_{i\in\mathcal{I}_q}C_{i,q}e^{-\xi_{i,q}x}\sum_{j=1}^{m}\frac{p_j}{\eta_j-\xi_{i,q}}(e^{-\eta_j\underline{a}}-e^{-\eta_j\overline{a}})\left(e^{-(\eta_j-\xi_{i,q})(\underline{b}\wedge x)}-e^{-(\eta_j-\xi_{i,q})(\bar{b}\wedge x)}\right).\end{aligned}$$

Hence

$$\begin{aligned}h_q(x;A,B)&=\frac{\lambda}{\psi'(\zeta_q)}e^{\zeta_qx}\sum_{j=1}^{m}\frac{p_j}{\eta_j+\zeta_q}(e^{-\eta_j\underline{a}}-e^{-\eta_j\overline{a}})\left(e^{-(\eta_j+\zeta_q)(\underline{b}\vee x)}-e^{-(\eta_j+\zeta_q)(\bar{b}\vee x)}\right)\\&+\lambda\sum_{i\in\mathcal{I}_q}C_{i,q}e^{-\xi_{i,q}x}\sum_{j=1}^{m}\frac{p_j}{\eta_j-\xi_{i,q}}(e^{-\eta_j\underline{a}}-e^{-\eta_j\overline{a}})\left(e^{-(\eta_j-\xi_{i,q})(\underline{b}\wedge x)}-e^{-(\eta_j-\xi_{i,q})(\bar{b}\wedge x)}\right)\\&-\lambda\sum_{i\in\mathcal{I}_q}C_{i,q}e^{-\xi_{i,q}x}\sum_{j=1}^{m}\frac{p_j}{\eta_j+\zeta_q}(e^{-\eta_j\underline{a}}-e^{-\eta_j\overline{a}})\left(e^{-(\eta_j+\zeta_q)\underline{b}}-e^{-(\eta_j+\zeta_q)\bar{b}}\right)\\&=\lambda\sum_{j=1}^{m}p_j(e^{-\eta_j\underline{a}}-e^{-\eta_j\overline{a}})\kappa_{j,q}(x;B).\end{aligned}$$

A.4. **Proof of Proposition 3.2.** Notice, for every $m \geq 1$, that

$$\begin{aligned} 0 &\leq \sum_{i=1}^{m} \left(C_{i,q} - C_{i,q}^{(m)} \right) \leq \gamma_q - \sum_{i=1}^{m} C_{i,q}^{(m)} = \delta_m, \\ 0 &\leq \sum_{i=m+1}^{\infty} C_{i,q} = \gamma_q - \sum_{i=1}^{m} C_{i,q} \leq \gamma_q - \sum_{i=1}^{m} C_{i,q}^{(m)} = \delta_m, \end{aligned} \tag{A.5}$$

and hence by (A.5)

$$\begin{aligned} 0 \leq \overline{W}_{\zeta_q}^{(m)}(x) - W_{\zeta_q}(x) &= \sum_{i=1}^{m} \left(C_{i,q} - C_{i,q}^{(m)} \right) e^{-(\zeta_q + \xi_{i,q})x} + \sum_{i=m+1}^{\infty} C_{i,q} e^{-(\zeta_q + \xi_{i,q})x} \\ &\leq e^{-\zeta_q x} \sum_{i=1}^{m} \left(C_{i,q} - C_{i,q}^{(m)} \right) + e^{-(\zeta_q + \xi_{m+1,q})x} \sum_{i=m+1}^{\infty} C_{i,q} \leq \delta_m \left[e^{-\zeta_q x} + e^{-(\zeta_q + \xi_{m+1,q})x} \right]. \end{aligned}$$

Therefore we have the bounds for W_{ζ_q} in (3.8). The bounds for $W^{(q)}$ are immediate by multiplying $e^{\zeta_q x}$. Finally, the convergence results hold because $\delta_m \to 0$ and $e^{-\zeta_q x} + e^{-(\zeta_q + \xi_{m+1,q})x}$ and $1 + e^{-\xi_{m+1,q} x}$ are bounded uniformly in $x \geq 0$.

A.5. **Proof of Proposition 3.3.** The lower bound is immediate by the fact that $0 \leq C_{i,q}^{(m)} \leq C_{i,q}$. For every $x > 0$, we have by (A.5)

$$\begin{aligned} W^{(q)\prime}(x) &= (\psi'(\zeta_q))^{-1} \zeta_q e^{\zeta_q x} + \sum_{i=1}^{\infty} C_{i,q} \xi_{i,q} e^{-\xi_{i,q} x} \\ &= \underline{w}^{(m)}(x) + \sum_{i=1}^{m} (C_{i,q} - C_{i,q}^{(m)}) \xi_{i,q} e^{-\xi_{i,q} x} + \sum_{i=m+1}^{\infty} C_{i,q} \xi_{i,q} e^{-\xi_{i,q} x} \\ &\leq \underline{w}^{(m)}(x) + \max_{1 \leq k \leq m} (\xi_{k,q} e^{-\xi_{k,q} x}) \sum_{i=1}^{m} (C_{i,q} - C_{i,q}^{(m)}) + \max_{k \geq m+1} (\xi_{k,q} e^{-\xi_{k,q} x}) \sum_{i=m+1}^{\infty} C_{i,q} \\ &\leq \underline{w}^{(m)}(x) + \left[\max_{1 \leq k \leq m} (\xi_{k,q} e^{-\xi_{k,q} x}) + \max_{k \geq m+1} (\xi_{k,q} e^{-\xi_{k,q} x}) \right] \delta_m = \overline{w}^{(m)}(x), \end{aligned}$$

which shows the first claim. For the second claim, notice that for the given $x_0 > 0$,

$$0 \leq \overline{w}^{(m)}(x) - \underline{w}^{(m)}(x) \leq \left[\max_{1 \leq k \leq m} (\xi_{k,q} e^{-\xi_{k,q} x_0}) + \max_{k \geq m+1} (\xi_{k,q} e^{-\xi_{k,q} x_0}) \right] \delta_m$$

uniformly on $[x_0, \infty)$. Because δ_m vanishes as $m \to 0$ and $\sup_{\lambda \geq 0} \lambda e^{-\lambda x} < \infty$, the convergence is immediate.

A.6. **Proof of Corollary 3.2.** For every $x > 0$,

$$\begin{aligned} W^{(q)\prime}(x) &= \underline{w}^{(m)}(x) + \sum_{i=1}^{m} (C_{i,q} - C_{i,q}^{(m)}) \xi_{i,q} e^{-\xi_{i,q} x} + \sum_{i=m+1}^{\infty} C_{i,q} \xi_{i,q} e^{-\xi_{i,q} x} \\ &\leq \underline{w}^{(m)}(x) + \max_{1 \leq k \leq m} (\xi_{k,q} e^{-\xi_{k,q} x}) \sum_{i=1}^{m} (C_{i,q} - C_{i,q}^{(m)}) + \frac{\zeta_q}{q} e^{-\xi_{m+1,q} x} \sum_{i=m+1}^{\infty} A_{i,q} \xi_{i,q} \\ &\leq \underline{w}^{(m)}(x) + \max_{1 \leq k \leq m} (\xi_{k,q} e^{-\xi_{k,q} x}) \delta_m + \epsilon_m e^{-\xi_{m+1,q} x} \end{aligned}$$

where the last inequality holds because

$$\frac{\zeta_q}{q}\sum_{k=m+1}^{\infty}\xi_{i,q}A_{i,q}=\theta-\frac{\zeta_q}{q}\sum_{k=1}^{m}\xi_{i,q}A_{i,q}\leq\theta-\frac{\zeta_q}{q}\sum_{k=1}^{m}\xi_{i,q}A_{i,q}^{(m)}=\epsilon_m,\quad m\geq 1.$$

This together with Proposition 3.3 shows the claim.

REFERENCES

[1] S. Asmussen. Fitting phase-type distributions via the EM algorithm. *Scand. J. Statist*, 23:419–441, 1996.

[2] S. Asmussen. *Applied probability and queues*, volume 51 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 2003. Stochastic Modelling and Applied Probability.

[3] S. Asmussen, F. Avram, and M. R. Pistorius. Russian and American put options under exponential phase-type Lévy models. *Stochastic Process. Appl.*, 109(1):79–111, 2004.

[4] S. Asmussen, D. B. Madan, and M. R. Pistorius. Pricing equity default swaps under an approximation to the CGMY Lévy model. *Journal of Computational Finance*, 11(2):79–93, 2007.

[5] F. Avram, Z. Palmowski, and M. R. Pistorius. On the optimal dividend problem for a spectrally negative Lévy process. *Ann. Appl. Probab.*, 17(1):156–180, 2007.

[6] O. E. Barndorff-Nielsen. Processes of normal inverse Gaussian type. *Finance Stoch.*, 2(1):41–68, 1998.

[7] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[8] J. Bertoin and M. Yor. On subordinators, self-similar Markov processes and some factorizations of the exponential variable. *Electron. Comm. Probab.*, 6:95–106 (electronic), 2001.

[9] M. Caballero, J. Pardo, and J. Pérez. On Lamperti stable processes. *Preprint*, 2008.

[10] M. E. Caballero and L. Chaumont. Conditioned stable Lévy processes and the Lamperti representation. *J. Appl. Probab.*, 43(4):967–983, 2006.

[11] P. Carr, G. Hélyette, D. B. Madan, and M. Yor. The structure of asset returns: an emperical investigation. *Journal of Business*, 75:305–332, 2002.

[12] T. Chan, A. E. Kyprianou, and M. Savov. Smoothness of scale functions for spectrally negative Lévy processes. *Probability Theory and Related Fields*, 2009.

[13] L. Chaumont, A. E. Kyprianou, and J. C. Pardo. Some explicit identities associated with positive self-similar Markov processes. *Stochastic Process. Appl.*, 119(3):980–1000, 2009.

[14] E. Eberlein, U. Keller, and K. Prause. New insights into smile, mispricing and value at risk: the hyperbolic model. *Journal of Business*, 71:371–405, 1998.

[15] A. Feldmann and W. Whitt. Fitting mixtures of exponentials to long-tail distributions to analyze network performance models. *Performance Evaluation*, (31):245–279, 1998.

[16] W. Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons Inc., New York, 1971.

[17] A. Horváth and M. Telek. Approximating heavy tailed behaviour with phase type distributions. *Advances in Algorithmic Methods for Stochastic Models*, 2000.

[18] J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2003.

[19] N. L. Johnson and S. Kotz. *Distributions in statistics: continuous multivariate distributions*. John Wiley & Sons Inc., New York, 1972. Wiley Series in Probability and Mathematical Statistics.

[20] A. Kuznetsov. Wiener-Hopf factorization and distribution of extrema for a family of Lévy processes. *Ann. Appl. Probab.*, 2009.

[21] A. Kuznetsov. Wiener-Hopf factorization for a family of Lévy processes related to theta functions. *Preprint*, 2009.

[22] A. Kuznetsov, A. E. Kyprianou, and J. C. Pardo. Meromorphic Lévy processes and their fluctuation identities. *Ann. Appl. Probab.*, to appear.

[23] A. Kuznetsov, A. E. Kyprianou, J. C. Pardo, and K. v. Schaik. A Wiener-Hopf Monte-Carlo simulation technique for Lévy processes. *Preprint*, 2010.

[24] A. Kuznetsov, A. E. Kyprianou, and V. Rivero. The theory of scale functions for spectrally negative Lévy processes. *Preprint*, 2011.

[25] A. E. Kyprianou. *Introductory lectures on fluctuations of Lévy processes with applications*. Universitext. Springer-Verlag, Berlin, 2006.

[26] A. E. Kyprianou. Exact and asymptotic n-tuple laws at first and last passage. *Ann. Appl. Probab.*, 20(2):522–564, 2010.

[27] A. E. Kyprianou and Z. Palmowski. Distributional study of de Finetti's dividend problem for a general Lévy insurance risk process. *J. Appl. Probab.*, 44(2):428–443, 2007.

[28] A. E. Kyprianou and V. Rivero. Special, conjugate and complete scale functions for spectrally negative Lévy processes. *Electron. J. Probab.*, 13:no. 57, 1672–1701, 2008.

[29] A. E. Kyprianou and B. A. Surya. Principles of smooth and continuous fit in the determination of endogenous bankruptcy levels. *Finance Stoch.*, 11(1):131–152, 2007.

[30] A. E. Kyprianou and X. Zhou. General tax structures and the Lévy insurance risk model. *J. Appl. Probab.*, 46(4):1146–1156, 2009.

[31] J. Lamperti. Semi-stable Markov processes. I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 22:205–225, 1972.

[32] A. Lang and J. L. Arthur. *Parameter Approximation for Phase-Type Distributions*. Matrix-analytic Methods in Stochastic Models (Lecture Notes in Pure and Applied Mathematics). CRC Press, 1996.

[33] R. L. Loeffen. On optimality of the barrier strategy in de Finetti's dividend problem for spectrally negative Lévy processes. *Ann. Appl. Probab.*, 18(5):1669–1680, 2008.

[34] R. L. Loeffen. An optimal dividends problem with a terminal value for spectrally negative Lévy processes with a completely monotone jump density. *J. Appl. Probab.*, 46(1):85–98, 2009.

[35] D. B. Madan, P. P. Carr, and E. C. Chang. The variance gamma processes and option pricing. *European Finance Review*, 2:79–105, 1998.

[36] D. B. Madan and F. Milne. Option pricing with VG martingale components. *Mathematical Finance*, 1(4):39–55, 1991.

[37] M. R. Pistorius. On maxima and ladder processes for a dense class of Lévy process. *J. Appl. Probab.*, 43(1):208–220, 2006.

[38] B. A. Surya. Evaluating scale functions of spectrally negative Lévy processes. *J. Appl. Probab.*, 45(1):135–149, 2008.

(M. Egami) GRADUATE SCHOOL OF ECONOMICS, KYOTO UNIVERSITY, SAKYO-KU, KYOTO, 606-8501, JAPAN
E-mail address: egami@econ.kyoto-u.ac.jp

(K. Yamazaki) CENTER FOR THE STUDY OF FINANCE AND INSURANCE, OSAKA UNIVERSITY, 1-3 MACHIKANEYAMA-CHO, TOYONAKA CITY, OSAKA 560-8531, JAPAN
E-mail address: k-yamazaki@sigmath.es.osaka-u.ac.jp